EconS 301- Intermediate Microeconomic Theory
Homework #4 - Due date: Thursday March 26th, 2026.

1. Sarah is looking into producing her homemade dog treats on a larger scale and is contemplating two different kitchen sizes (K). Her production of dog treats follows $q = 200KL + K^2L^3$. What are the marginal and average product curves for labor when $K = 5$? What happens to the marginal and average product curves when her kitchen doubles to $K = 10$?

   - The marginal product curve is found by differentiating her production function with respect to L:
   $$MP_L = \frac{\partial q}{\partial L} = 200K + 3K^2L^2.$$
   The average product curve is found by dividing the production function by labor:
   $$AP_L = \frac{q}{L} = \frac{200KL}{L} + \frac{K^2L^3}{L} = 200K + K^2L^2.$$
   At $K = 5$ units of capital,
   $$MP_L = 200(5) + 3(5)^2L^2 = 1000 + 75L^2, \text{ and}$$
   $$AP_L = 200(5) + (5)^2L^2 = 1000 + 25L^2.$$
   At $K = 10$ units of capital,
   $$MP_L = 200(10) + 3(10)^2L^2 = 2000 + 300L^2, \text{ and}$$
   $$AP_L = 200(10) + (10)^2L^2 = 2000 + 100L^2.$$

2. Is it possible for two firms with different Cobb-Douglas production functions, $q_1 = AL^\alpha K^\beta$ and $q_2 = L^\alpha K^\beta$, to have the same marginal products at particular levels of capital and labor? What about each firms' marginal rate of technical substitution.

   - The marginal product of labor for the first production function, q_1, is
   $$MP_{L1} = \frac{\partial q_1}{\partial L} = \alpha AL^{\alpha-1}K^\beta,$$
   and for the second production function, q_2, is
   $$MP_{L2} = \frac{\partial q_1}{\partial L} = \alpha L^{\alpha-1}K^\beta$$
   Comparing we see that $MP_{L1} = MP_{L2}$ if
   $$\alpha AL^\alpha K^\beta = \alpha L^\alpha K^\beta,$$
   which happens when $A = 1$. Therefore, the only time the marginal products will be equal is when they are the same production function. A similar argument applies for the marginal product of capital.

However, the MRTS between the two will always coincide. To see this,

$$\begin{aligned}
MRTS_{q1} = \frac{\alpha AL^{\alpha-1}K^{\beta}}{\beta AL^{\alpha}K^{\beta-1}} &= \frac{\alpha L^{\alpha-1}K^{\beta}}{\beta L^{\alpha}K^{\beta-1}} = MRTS_{q2} \\
\frac{\alpha L^{\alpha-1}K^{\beta}}{\beta L^{\alpha}K^{\beta-1}} &= \frac{\alpha L^{\alpha-1}K^{\beta}}{\beta L^{\alpha}K^{\beta-1}} \\
\frac{\alpha K}{\beta L} &= \frac{\alpha K}{\beta L},
\end{aligned}$$

which coincides for all levels of capital and labor.

3. A firm has the following cost function:

$$TC(q) = 2q^3 - \frac{1}{3}q^2 + \frac{1}{2}q + \frac{9}{10},$$

where q denotes units of output. Intuitively, the first three terms on the right side capture the firm's variable cost, because they depend on the output the firm produces, whereas the last term represents its fixed cost, as it is not a function of output q.

(a) *Total cost.* For which output q does the total cost curve $TC(q)$ increase or decrease? For which values is it concave or convex in output?

- We need to look at the derivative:

$$\frac{dC}{dq} = 6q^2 - \frac{2}{3}q + \frac{1}{2}.$$

This is always greater than 0, which means that the total cost curve is always increasing. To evaluate the concavity, we look at the second derivative:

$$\frac{d^2C}{dq^2} = 12q - \frac{2}{3}.$$

Setting equal to zero and solving for q, we can find that the second derivative is positive for $q > \frac{1}{18}$, which is where the total cost curve is convex. For $q < \frac{1}{18}$, the total cost curve is concave.

(b) *Marginal cost.* For which output q does the marginal cost curve $\frac{\partial TC(q)}{\partial q}$ increase or decrease? For which values is it concave or convex in output?

- We found marginal cost in the last problem, $MC = 6q^2 - \frac{2}{3}q + \frac{1}{2}$, and its derivative, $12q - \frac{2}{3}$, which tells us that marginal cost is increasing for $q > \frac{1}{18}$. The second derivative of marginal cost, $\frac{d^2MC}{dq^2} = 12 > 0$ means that the marginal cost curve is convex.

(c) *Average cost.* For which output q does the average cost curve $AC(q) = \frac{TC(q)}{q}$ increase or decrease? For which values is it concave or convex in output?

- Average cost is

$$\frac{C(q)}{q} = 2q^2 - \frac{1}{3}q + \frac{1}{2} + \frac{9}{10q}.$$

The derivative of average cost is

$$\frac{dAC}{dq} = 4q - \frac{1}{3} - \frac{9}{10q^2}.$$

Solving for when this equals zero, we find that the average cost is decreasing for $q < 0.52$ but increasing for $q > 0.52$. The second derivative tells us about its concavity:

$$\frac{d^2AC}{dq^2} = 4 + \frac{18}{10q},$$

which is always positive, so average cost is convex.

(d) *Average variable cost.* For which output q does the average variable cost curve $AC(q)$ increase or decrease? For which values is it concave or convex in output?

- The variable cost is part of costs that varies with output: $2q^3 - \frac{1}{3}q^2 + \frac{1}{2}q$. Therefore, average variable cost is:

$$AVC = 2q^2 - \frac{1}{3}q + \frac{1}{2}.$$

The derivative will tell us when it increases or decreases:

$$\frac{dAVC}{dq} = 4q - \frac{1}{3}.$$

Setting equal to zero and solving, we find that average variable cost decreases when $q < \frac{1}{12} \simeq 0.083$, and increases at larger values of q. The second derivative, $\frac{d^2AVC}{dq^2} = 4$, tells us that the average variable cost is convex.

(e) Find the value of q where the marginal cost curve crosses the total cost curve, where it crosses the average cost curve, and where it crosses the average variable cost curve.

- The marginal cost curve crosses the average cost and average variable cost curves at their lowest points, which we have already calculated. Marginal and average costs cross at $q = 0.637$, while marginal and average variable costs cross at $q = 0.083$.

4. Consider a firm with the Cobb-Douglas production function $f(K, L) = 4K^{1/2}L^{1/3}$, where K denotes units of capital and L represents units of labor. Assume that the firm faces input prices of $r = \$10$ per unit of capital, and $w = \$7$ per unit of labor.

(a) Solve the firm's cost-minimization problem, to obtain the combination of inputs (labor and capital) that minimizes the firm's cost of producing a given amount of output, q.

- The tangency condition $\frac{MP_L}{MP_K} = \frac{w}{r}$ is

$$\frac{\frac{4}{3}K^{1/2}L^{-2/3}}{2K^{-1/2}L^{1/3}} = \frac{7}{10}$$

which simplifies to

$$\frac{2}{3}\frac{K}{L} = \frac{7}{10}$$

This contains both L and K, so we solve for K:

$$K = \frac{7}{10}\frac{3}{2}L = \frac{21}{20}L$$

We can insert this back into the firm's output target $q = 4K^{1/2}L^{1/3}$, to find that

$$q = 4\underbrace{\left(\frac{21}{20}L\right)}_{K}^{1/2} L^{1/3}$$

rearranging,

$$q = 4\left(\frac{21}{20}\right)^{1/2} L^{2/6+3/6},$$

where we need to solve for L:

$$\begin{aligned} L^{5/6} &= \frac{q}{4}\left(\frac{20}{21}\right)^{1/2} \\ L &= \left(\frac{q}{4}\left(\frac{20}{21}\right)^{1/2}\right)^{6/5} \\ L &= \left(\frac{q}{4}\right)^{6/5}\left(\frac{20}{21}\right)^{3/5} \simeq 0.184q^{6/5}. \end{aligned}$$

This is the firm's labor demand. Plugging this back into the tangency condition,

$$K = \frac{21}{20}\left(\frac{q}{4}\right)^{6/5}\left(\frac{20}{21}\right)^{3/5}$$

which simplifies to

$$K = \left(\frac{q}{4}\right)^{6/5}\left(\frac{21}{20}\right)^{2/5} \simeq 0.193q^{6/5}$$

which is the firm's demand for capital.

(b) Use your results from part (a) to find the firm's cost function. This is its long-run total cost, as all inputs can be altered.

- Plugging back into the cost function, we get

$$C = wL + rK = 7\underbrace{\left(\frac{q}{4}\right)^{6/5}\left(\frac{20}{21}\right)^{3/5}}_{L} + 10\underbrace{\left(\frac{q}{4}\right)^{6/5}\left(\frac{21}{20}\right)^{2/5}}_{K}.$$

If we simplify this numerically, we get

$$C \simeq 3.218q^{6/5}.$$

(c) Find the firm's marginal cost function, and its average cost function. Interpret.

- The firm's marginal cost is

$$MC = \frac{dC}{dq} = \frac{6}{5}3.218q^{1/5} \simeq 3.862q^{1/5}.$$

Each additional unit the firm produces will cost about $3.862q^{1/5}$, which increases as q increases.
The firm's average cost is

$$AC = \frac{C}{q} = \frac{3.218q^{6/5}}{q} = 3.218q^{1/5}.$$

This average cost lies below the marginal cost, but has the same shape. Figure 1 depicts marginal and average costs.

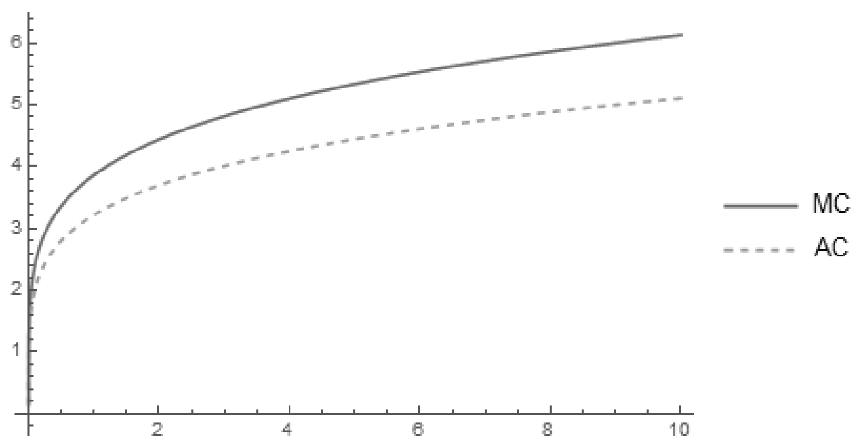

Figure 1. Marginal and average cost curves.

(d) Assume now that the amount of capital is held fixed at $\overline{K} = 3$ units. Solve the firm's cost-minimization problem again to find the amount of labor that minimizes the firm's cost.

- If $\overline{K} = 3$, the firm's cost-minimizing amount of labor satisfies

$$q = 4\underbrace{(3)}_{\overline{K}}{}^{1/2}L^{1/3}.$$

Solving for L, we find that

$$L^{1/3} = \frac{q}{4(3)^{1/2}}$$
$$L = \left(\frac{q}{4(3)^{1/2}}\right)^3$$
$$L = \frac{q^3}{64(3)^{3/2}} \simeq 0.003q^3.$$

(e) Use your results from part (d) to find the firm's short-run cost function (since in the short run the firm can only alter the amount of labor, but without changing the units of capital).

- Plugging the previous result into the firm's total cost, we get

$$C = 7\underbrace{(0.003q^3)}_{L} + 10\underbrace{(3)}_{\overline{K}} = 0.021q^3 + 30.$$