EconS 301- Intermediate Microeconomic Theory
Homework #1 - Due date: Tuesday January 27th, 2026.

1. Consider the following utility function $u(x,y) = ax + by$ and assume that $a = 2$ and $b = 3$. Show that this utility function satisfies completeness, transitivity, monotonicity, strict monotonicity, and non-satiation. Then consider a different utility function $u(x,y) = \min\{2x, 3y\}$ and show that the utility function satisfies all properties, except for strict monotonicity.

   - Starting withu utility function $u(x,y) = 2x + 3y$:
     - *Completeness.* Like in Example 2.4, as long as we can specify two bundles $A = (x_A, y_A)$ and $B = (x_B, y_B)$ and rank them such that either $u(x_A, y_A) \geq u(x_B, y_B)$, $u(x_B, y_B) \geq u(x_A, y_A)$, or both, completeness is satisfied. For any values of x and y, we obtain a real value for the utility function $u(x,y) = 2x + 3y$, so we can compare any pair of bundles by simply comparing the utility values. Thus, $u(x,y)$ satisfies completeness.
     - *Transitivity.* Again, for every three bundles A, B and C, where $u(x_A, y_A) \geq u(x_B, y_B)$ and $u(x_B, y_B) \geq u(x_C, y_C)$, we must have that $u(x_A, y_A) \geq u(x_C, y_C)$ for transitivity to hold. Just like we did for completeness, we can translate any pair of values for x and y into a corresponding utility value, then rank them accordingly. Thus, transitivity holds by definition.
     - *Strict Monotonicity.* An increase in either the value of x or y leads to a strictly higher value of utility. Thus, strict monotonicity holds.
     - *Monotonicity.* Since the utility function is strictly monotonic, it is also monotonic. Intuitively, if we increase both the values of x and y in the utility function, an individual receives a strictly higher utility level.
     - *Non-satiation.* Non-satiation is satisfied by monotonicity. If we increase a bundle from (x, y) to $(x + b, y + a)$ for any positive values of a and b (i.e., increasing the amount of good x by b units, and the amount of good y by a units), an individual receives a strictly higher utility level. Thus, the individual never reaches a bliss point and her preferences satisfy non-satiation.
   - For the new utility function, we follow a similar process.
     - *Completeness.* For completeness to hold, we need that, for any two bundles $A = (x_A, y_A)$ and $B = (x_B, y_B)$, we can rank them such that either $u(x_A, y_A) \geq u(x_B, y_B)$, $u(x_B, y_B) \geq u(x_A, y_A)$, or both. For any real values of x and y, we obtain a real value for the utility function $u(x,y) = \min\{2x, 3y\}$, and can compare any pair of bundles by simply comparing their utility values. Thus, $u(x,y)$ satisfies completeness.
     - *Transitivity.* Since we can translate any possible bundle into a numerical utility value, we can rank them appropriately. If bundle A is preferred to bundle B and bundle B is preferred to bundle C, then $u(x_A, y_A) > u(x_B, y_B)$ and $u(x_B, y_B) > u(x_C, y_C)$. This implies that $u(x_A, y_A) > u(x_C, y_C)$ and thus bundle A must be preferred to bundle C and $u(x,y)$ satisfies transitivity.

- *Strict Monotonicity.* the new utility function does not satisfy strict monotonicity. Suppose $x = 1$ and $y = 10$. John's utility level is

$$u(1, 10) = \min\{2(1), 3(10)\} = \min\{2, 30\} = 2.$$

  If we increase the amount of good y to 11 units, we obtain the same utility value:

$$u(1, 11) = \min\{2(1), 3(11)\} = \min\{2, 33\} = 2.$$

  Intuitively, an individual prefers to consume goods x and y in fixed proportions, and increasing an overly proportioned good (good y in this case) does not increase the individual utility. Since an increase of one component of the bundle does not necessarily lead to a more preferred bundle, the new utility function does not satisfy strict monotonicity.
- *Monotonicity.* If we increase both values of x and y in the new utility function, the lesser between $2x$ and $3y$ must increase as well. To see this point with our above example, consider bundle $(1, 10)$, which yields a utility level of

$$u(1, 10) = \min\{2(1), 3(10)\} = \min\{2, 30\} = 2,$$

  and bundle $(2, 11)$, where both goods increased by one unit, which yields a utility level

$$u(2, 11) = \min\{2(2), 3(11)\} = \min\{4, 33\} = 4.$$

  Thus, an individual reaches a strictly higher utility level and her utility function satisfies monotonicity.
- *Non-satiation.* Since the utility function satisfies monotonicity, it satisfies non-satiation. If we increase a bundle from (x, y) to $(x + b, y + a)$ for any positive values of a and b (i.e., increasing the amount of good x by b units, and the amount of good y by a units), the lesser of $2x$ and $3y$ must also increase and an individual receives a strictly higher utility level. Thus, an individual never reaches a bliss point and his preferences satisfy non-satiation.

2. Consider a consumer with utility function $u(x, y) = 3y + 2x$ who seeks to reach a utility level $u = 20$. Solve for y to find her indifference curve. Is it increasing or decreasing? What if her utility function is $u(x, y) = 3y - 2x$?

- *First utility function.* For a utility function of $u(x, y) = 3y + 2x$ and a utility level of $u = 20$, we know that his indifference curve entails $3y + 2x = 20$. We can find his indifference curve by solving this expression for y.
  - Subtracting both sides of this equation by $2x$ gives $3y = 20 - 2x$.
  - Next, we divide both sides by 3 to obtain his indifference curve,

$$y = \frac{20}{3} - \frac{2}{3}x.$$

This indifference curve is clearly decreasing in x, as depicted in figure 1.

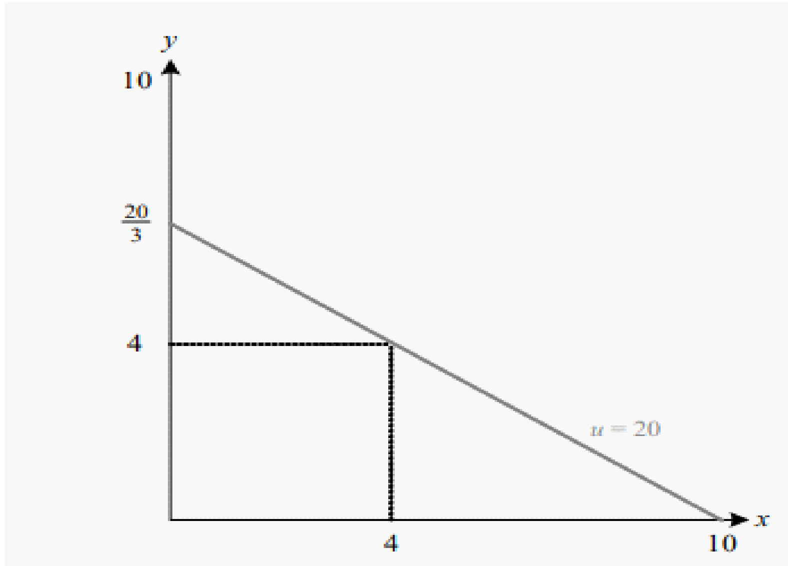

Figure 1. Indifference curve $y = \frac{20}{3} - \frac{2}{3}x$.

- *Second utility function.* For the second utility function of $u(x, y) = 3y - 2x$ and utility level of $u = 20$, we follow the same steps as before and solve for y to derive his indifference curve.
  - Adding $2x$ to both sides of this equation gives $3y = 20 + 2x$.
  - Last, we divide both sides by 3 to obtain the new indifference curve,

$$y = \frac{20}{3} + \frac{2}{3}x.$$

  Since this indifference curve has a positive slope, it is actually increasing in x, as depicted in figure 2.

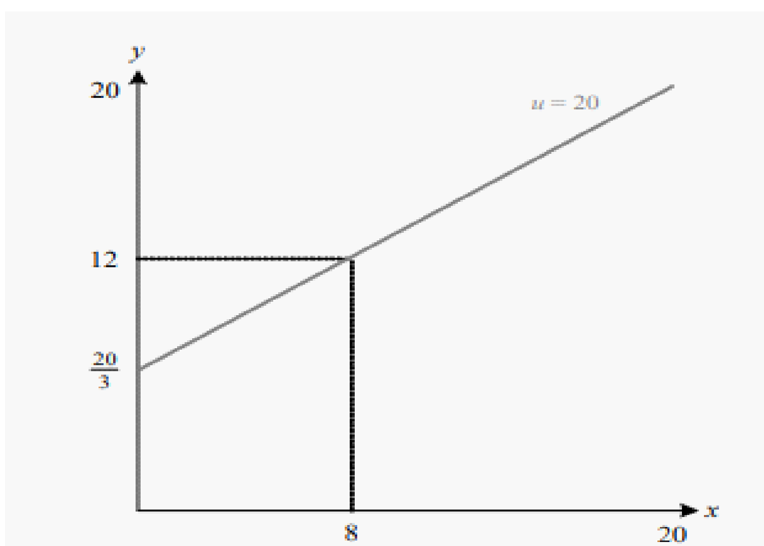

Figure 2. Indifference curve $y = \frac{20}{3} + \frac{2}{3}x$.

- This should make sense, as good x is a bad. In this case, our consumer would rather consume bundles with more of good y and less of good x. Intuitively, our consumer moves from a lower utility curve to a higher utility curve by moving upward and to the left. The leftward movement corresponds with good x's status as a bad. This effect is depicted in figure 3, as it shows the consumer's movement from $u = 20$ to a higher indifference curve at $u = 40$.

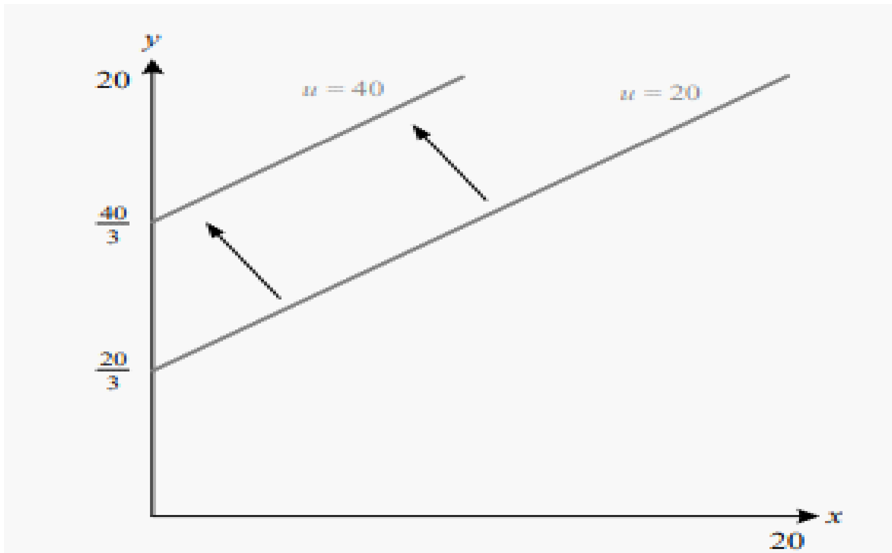

Figure 3. Indifference curves with x as a "bad."

3. Maria's utility function is $u(x, y) = 5x^{1/2}y^{1/4}$. Graph her indifference curve for utility levels $u = 10$ and $u = 20$. What is the utility elasticity of good x? And of good y? Interpret.

   - Since Maria's utility function falls under the Cobb-Douglas classification, we know that her indifference curves are not linear, but are rather bowed-in toward the origin. As before, we can solve for y in Maria's utility function to find her indifference curves.
     - First, we divide both sides of $u(x, y) = 5x^{1/2}y^{1/4}$ by $5x^{1/2}$ to obtain

$$y^{1/4} = \frac{u}{5x^{1/2}}.$$

     - Next we raise both sides of this equation to the fourth power to obtain our equation for Maria's indifference curves,

$$y = \frac{u^4}{625x^2}.$$

     - Plugging in our different values for u gives us two different indifference curves,

$$\begin{aligned} y &= \frac{10,000}{625x^2} \text{ for } u = 10, \text{ and} \\ y &= \frac{160,000}{625x^2} \text{ for } u = 20. \end{aligned}$$

– We can plot these indifference curves by examining a few bundles that fall on these indifference curves:

  * For $u = 10$, if we set $x = 2$, we have that $y = 4$ units. Likewise, when $x = 4$, good y becomes $y = 1$ units.
  * For $u = 20$, if we set $x = 4$, good y is $y = 16$ units. When $x = 8$, good y becomes $y = 4$ units.
  * These bundles allow us to find a few relevant points along our indifference curves, which are depicted in figure 4.

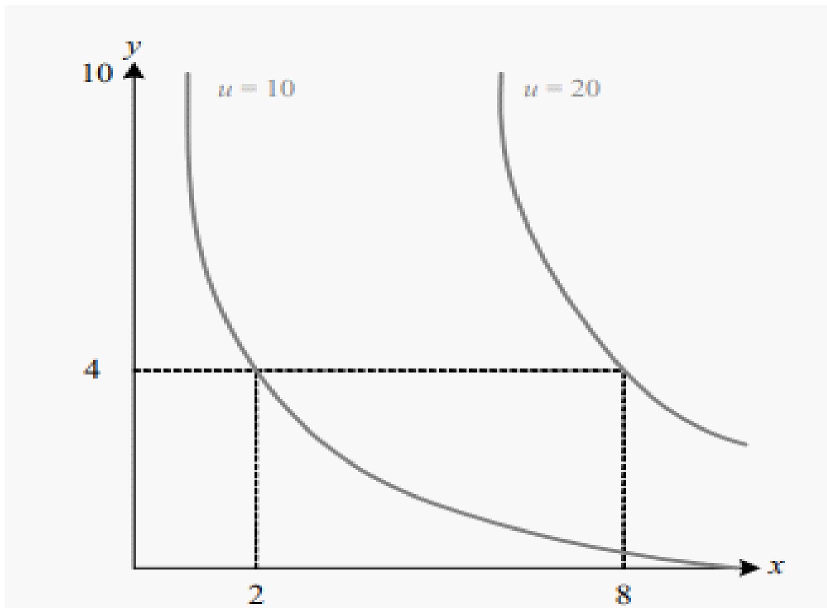

Figure 4. Maria's indifference curves.

- *Utility elasticity of good x.* We can calculate Maria's utility elasticity of good x from the formula

$$\begin{aligned}\varepsilon_{u,x} &= \frac{\partial u(x,y)}{\partial x}\frac{x}{u(x,y)}\\ &= 5\frac{1}{2}x^{-1/2}y^{1/4}\frac{x}{5x^{1/2}y^{1/4}}\\ &= \frac{1}{2}\frac{y^{\frac{1}{4}-\left(\frac{1}{4}\right)}}{x^{\frac{1}{2}-\left(\frac{1}{2}\right)}}\\ &= \frac{1}{2}.\end{aligned}$$

  This implies that a 1 percent increase in good x yields a 0.5 percent increase in Maria's utility.

- *Utility elasticity of good y.* Using the same process to calculate Maria's utility elasticity of good y, we obtain

$$\begin{aligned}
\varepsilon_{y,x} &= \frac{\partial u(x,y)}{\partial y}\frac{y}{u(x,y)} \\
&= 5\frac{1}{4}x^{1/2}y^{-3/4}\frac{y}{5x^{1/2}y^{1/4}} \\
&= \frac{1}{4}\frac{y^{-\frac{3}{4}+1-\frac{1}{4}}}{x^{\frac{1}{2}-\left(\frac{1}{2}\right)}} \\
&= \frac{1}{4}.
\end{aligned}$$

In words, this result says that a 1 percent increase in good y corresponds with a 0.25 percent increase in Maria's utility.

4. Answer the following questions for each of the utility functions in Table 2.1 (Chapter 2, page 17).

   (a) Find the marginal utility for good x and y, MU_x and MU_y.

   - $u(x,y) = by$:

   $$MU_x = \frac{\partial u(x,y)}{\partial x} = 0 \quad \text{and} \quad MU_y = \frac{\partial u(x,y)}{\partial y} = b.$$

   Therefore, this consumer enjoys a constant and positive utility from additional units of good y, $MU_y = b$. (Recall that parameter b was assumed to be positive for all utility functions in table 2.1.) In contrast, the consumer does not care about the units of good x she has since her marginal utility is zero, $MU_x = 0$.

   - $u(x,y) = ax$:

   $$MU_x = \frac{\partial u(x,y)}{\partial x} = a \quad \text{and} \quad MU_y = \frac{\partial u(x,y)}{\partial y} = 0.$$

   Therefore, this consumer enjoys a constant and positive utility from additional units of good x, $MU_x = a$. (Recall that parameter a was assumed to be positive for all utility functions in table 2.1.) In contrast, the consumer does not care about the units of good y she has since her marginal utility is zero, $MU_y = 0$.

   - $u(x,y) = ax - by$:

   $$MU_x = \frac{\partial u(x,y)}{\partial x} = a \quad \text{and} \quad MU_y = \frac{\partial u(x,y)}{\partial y} = -b.$$

   Therefore, this consumer enjoys a constant and positive utility from additional units of good x, $MU_x = a$. (Recall that parameters a and b were assumed to be positive for all utility functions in table 2.1.) In contrast, the consumer receives a constant and negative utility from additional units of good y, $MU_y = -b$, which implies that good y is a bad.

- $u(x, y) = ax + by$:

$$MU_x = \frac{\partial u(x,y)}{\partial x} = a \quad \text{and} \quad MU_y = \frac{\partial u(x,y)}{\partial y} = b.$$

  Therefore, this consumer enjoys a constant and positive utility from additional units of both goods x and y, $MU_x = a$ and $MU_y = b$, respectively. (Recall that parameters a and b were assumed to be positive for all utility functions in table 2.1.)

- $u(x, y) = A \min\{ax, by\}$:

$$\begin{aligned} MU_x &= \frac{\partial u(x,y)}{\partial x} = Aa \text{ if } ax < by, \text{ but } MU_x = 0 \text{ otherwise, and} \\ MU_y &= \frac{\partial u(x,y)}{\partial y} = Ab \text{ if } ax > by, \text{ but } MU_y = 0 \text{ otherwise.} \end{aligned}$$

  Therefore, this consumer only receives utility from the good that they have less of. When the consumer has relatively less of good x than good y (i.e., when $ax < by$), the consumer enjoys a constant and positive utility from additional units of good x, $MU_x = Aa$, and zero utility from additional units of good y, $MU_y = 0$. For example, if x represents left shoes and y denotes right shoes, and the consumer has more units of left shoes (x), then increasing the number of right shoes (y) can provide her with more pairs of shoes, thus increasing her utility.

  On the contrary, when the consumer has relatively less of good y than good x (i.e., when $ax > by$), the consumer enjoys a constant and positive utility from additional units of good y, $MU_y = Ab$, and zero utility from additional units of good x, $MU_x = 0$.

- $u(x, y) = Ax^{\alpha}y^{\beta}$:

$$MU_x = \frac{\partial u(x,y)}{\partial x} = \alpha A x^{\alpha-1} y^{\beta} \quad \text{and} \quad MU_y = \frac{\partial u(x,y)}{\partial y} = \beta A x^{\alpha} y^{\beta-1}.$$

  Therefore, this consumer enjoys a positive utility from additional units of both goods x and y, but the rate at which the utility increases changes depending on the relative consumption levels of both goods.

(b) Are these marginal utilities positive? Are they strictly positive? Connect your results with the properties of monotonicity and strict monotonicity.

- $u(x, y) = by$: In this case, only the marginal utility with respect to y is positive. Since the marginal utility of x is not positive, this utility function is monotonic, but not strictly monotonic; in fact, it is constant in x, implying that the indifference curve is straight (constant slope).
- $u(x, y) = ax$: In this case, only the marginal utility with respect to x is positive. Since the marginal utility of y is not positive, this utility function is monotonic, but not strictly monotonic; in fact, it is constant in x, implying that the indifference curve is straight (constant slope).

- $u(x, y) = ax - by$: In this case, the marginal utility with respect to y is negative. Thus, this utility function is neither monotonic nor strictly monotonic; in fact, it is constant in x, implying that the indifference curve is straight (constant slope).
- $u(x, y) = ax + by$: Since both marginal utilities are positive, this utility function is both monotonic and strictly monotonic; in fact, it is constant in x, implying that the indifference curve is straight (constant slope).
- $u(x, y) = A\min\{ax, by\}$: In this case, the marginal utilities for x and y are not strictly positive, but they are never negative, and one is always positive. Thus, this utility function is monotonic, but not strictly monotonic.
- $u(x, y) = Ax^{\alpha}y^{\beta}$: Since both marginal utilities are positive, this utility function is both monotonic and strictly monotonic.

(c) Find $MRS = \frac{MU_x}{MU_y}$. Does MRS increase in the amount of good x?

- $u(x, y) = by$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{0}{b} = 0.$$

Therefore, differentiating $MRS_{x,y}$ with respect to x, we obtain

$$\frac{\partial MRS}{\partial x} = 0.$$

The marginal rate of substitution is not increasing in x.

- $u(x, y) = ax$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{a}{0}.$$

Therefore, the marginal rate of substitution is not increasing in x.

- $u(x, y) = ax - by$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{a}{-b} = -\frac{a}{b}.$$

Therefore, differentiating $MRS_{x,y}$ with respect to x, we obtain

$$\frac{\partial MRS}{\partial x} = 0.$$

The marginal rate of substitution is not increasing in x.

- $u(x, y) = ax + by$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{a}{b}.$$

Therefore, differentiating $MRS_{x,y}$ with respect to x, we obtain

$$\frac{\partial MRS}{\partial x} = 0.$$

The marginal rate of substitution is not increasing in x.

- $u(x, y) = A \min\{ax, by\}$:
  The marginal rate of substitution is not well defined when we have perfect complements. Intuitively, the indifference curve is vertical for all points to the left of the kink, horizontal for all points to the right of the kink, but undefined at the kink since we can draw infinitely many tangent lines at the kink, each with a different slope.
- $u(x, y) = Ax^{\alpha}y^{\beta}$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{\alpha A x^{\alpha-1} y^{\beta}}{\beta A x^{\alpha} y^{\beta-1}} = \frac{\alpha y}{\beta x}.$$

  Therefore, differentiating $MRS_{x,y}$ with respect to x we obtain

$$\frac{\partial MRS}{\partial x} = -\frac{\alpha y}{\beta x^2} < 0.$$

  The marginal rate of substitution is decreasing in x.

(d) Depict an indifference curve reaching a utility level of $u = 10$ and another of $u = 20$. Do the indifference curves cross either axis?

- $u(x, y) = by$: In this case, the indifference curves take the form $y = \frac{u}{b}$, which is unaffected by the value of x. Graphically, this means that indifference curves cross the vertical axis, as illustrated in figure 5.

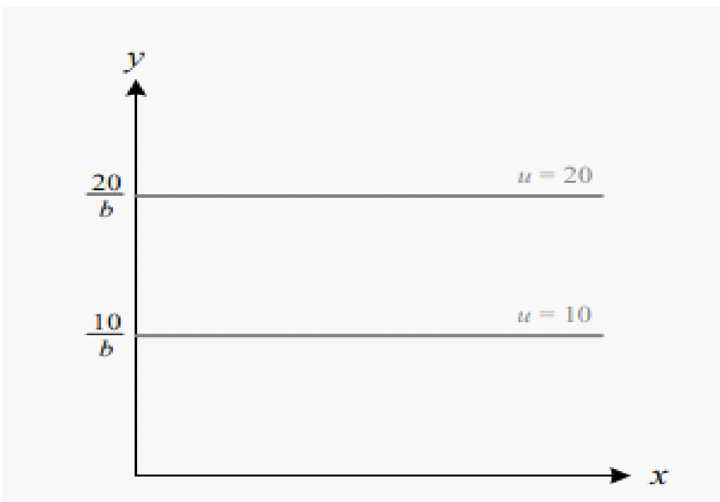

Figure 5. Horizontal indifference curves.

$u(x, y) = ax$: In this case, the indifference curves take the form $x = \frac{u}{a}$, which is unaffected by the value of y. Graphically, this means that indifference curves cross the horizontal axis of figure 6.

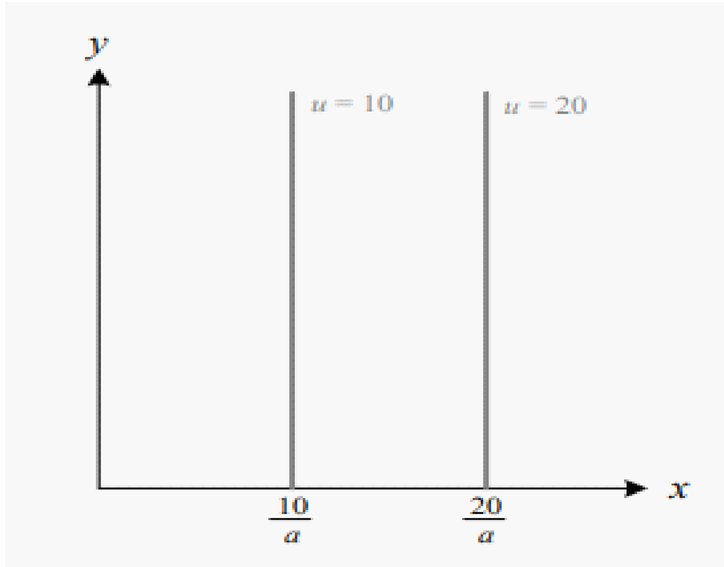

Figure 6. Vertical indifference curves.

- $u(x, y) = ax - by$: In this case, the indifference curve takes the form $y = \frac{u}{b} + \frac{a}{b}x$, which is increasing in the value of x. Intuitively, an increase in good x increases the individual's utility, so she needs to reduce her consumption of good y to maintain her utility unchanged. Graphically, this means that, as we move rightward (higher values of x in figure 7), we need to move downward (lower values of y) along a given indifference curve to keep the consumer's utility level unaffected.

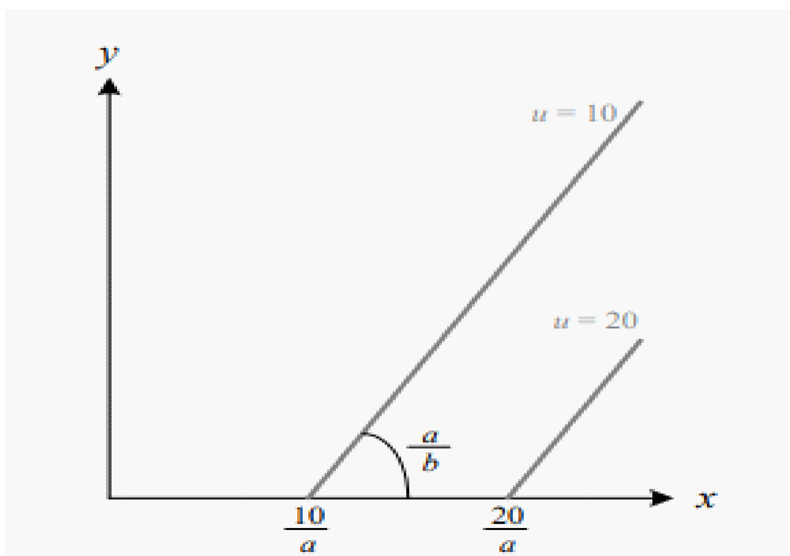

Figure 7. Perfectly "bad" substitutes.

- $u(x, y) = ax + by$: In this case, the indifference curve takes the form $y = \frac{u}{b} - \frac{a}{b}x$, which is decreasing in the value of x. Graphically, this means that indifference curves cross both axes. Intuitively, this consumer is completely indifferent with a proportional reduction of b units of good y in exchange for a units of good x. It does not matter how much of either good they already have; they will always be indifferent between trades of that proportion. Figure 8 depicts two indifferent curves: one evaluated at $u = 10$, so its vertical intercept becomes $y = \frac{10}{b}$, and another evaluated at $u = 20$, with vertical intercept $y = \frac{20}{b}$.

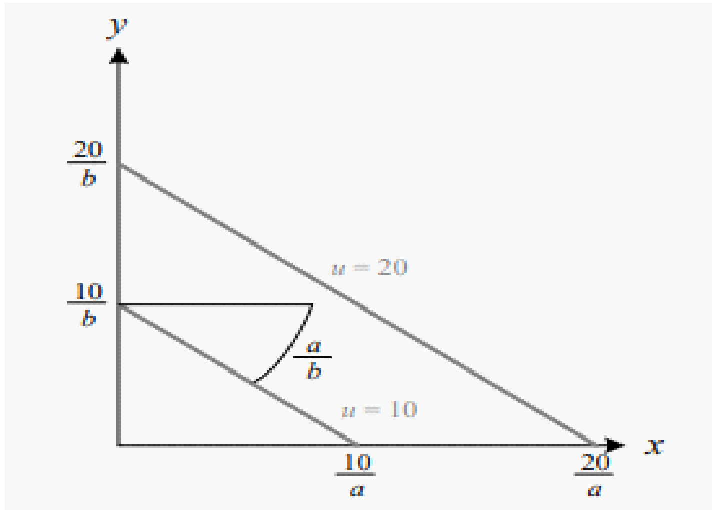

Figure 8. Perfect substitutes.

- $u(x, y) = A \min\{ax, by\}$: In this case, the indifference curves cross neither axis, as depicted in figure 9.

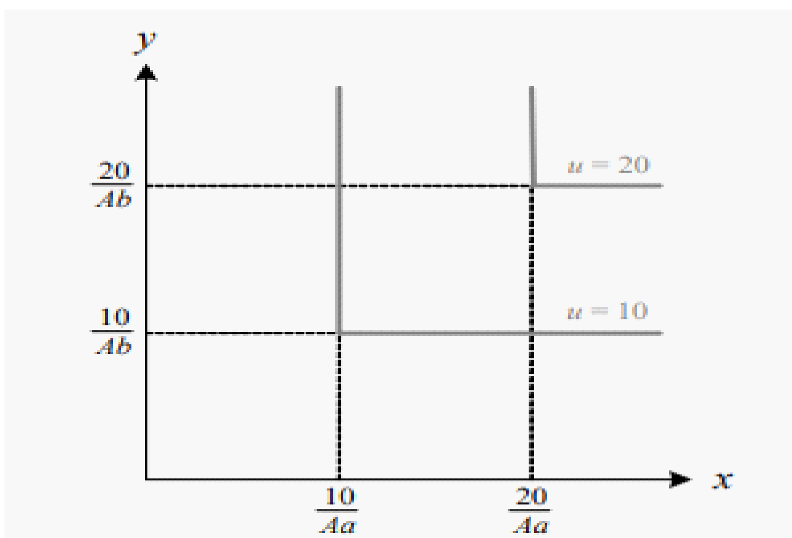

Figure 9. Perfect complements.

- $u(x, y) = Ax^{\alpha}y^{\beta}$: In this case, we can derive our indifference curves by solving this expression for y,

$$y = \left(\frac{u}{Ax^{\alpha}}\right)^{\frac{1}{\beta}}.$$

Intuitively, consumers' utility shifts outward as they consume more units of either good x or good y. They are also willing to trade units of good x for units of good y, but the ratio they require to remain indifferent increases as they have less of their respective good. In addition, the parameters α and β determine how far the indifference curves shift as the consumption level of both good increases, as well as the relative curvature of the indifference curve. Figure 10 plots this curve evaluated at $u = 10$ and $u = 20$ where $A = 1$, $\alpha = 0.5$, and $\beta = 0.5$.

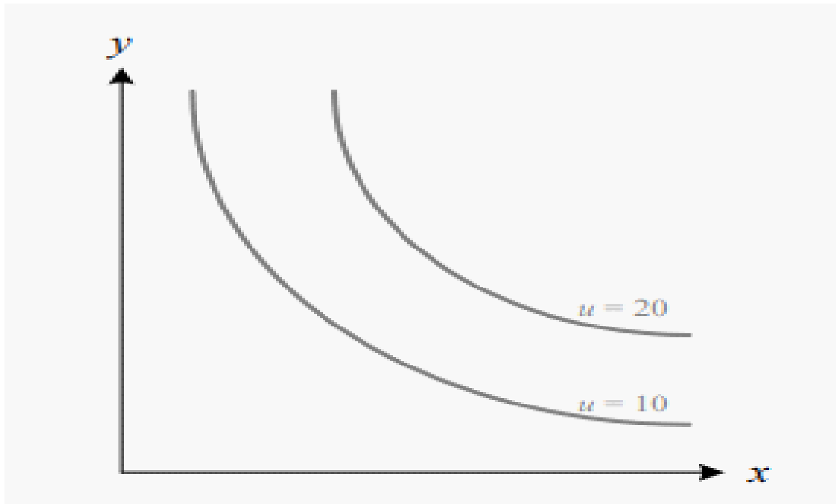

Figure 10. Cobb-Douglas preferences.

(e) Provide an example of goods that you think can be represented with each utility function in Table 2.1 (Chapter 2, page 17).

- $u(x, y) = by$: Good x could be air, whereas good y is water.
- $u(x, y) = ax$: Good x could be water, whereas good y is air.
- $u(x, y) = ax - by$: Good x could be water, whereas good y is garbage.
- $u(x, y) = ax + by$: Goods x and y could be different brands of milk.
- $u(x, y) = A\min\{ax, by\}$: Goods x and y could be left shoes and right shoes, respectively.

5. Find the marginal rate of substitution (MRS) for each of the utility functions. Are the MRS you found diminishing? Provide an economic interpretation for each MRS.

(a) $u(x, y) = by$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{0}{b} = 0$$

The marginal rate of substitution is not diminishing; in fact it is not impacted by good x at all. Since only good y impacts our utility function, our indifference curves are horizontal.

(b) $u(x, y) = ax$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{a}{0} = \infty$$

The marginal rate of substitution once again does not change with changes to x. This time, our indifference curves are vertical.

(c) $u(x,y) = ax - by$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{a}{-b} = -\frac{a}{b}$$

The marginal rate of substitution is not diminishing. The consumer always wants to consume more of good x and less of good y.

(d) $u(x,y) = ax + by$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{a}{b}$$

The marginal rate of substitution is not diminishing. The consumer values a units of good x as good as b units of good y, and never gets tired of either.

(e) $u(x,y) = A\min\{ax, by\}$: The marginal rate of substitution is not well defined when we have perfect complements. Intuitively, the indifference curve is vertical for all points above the kink, horizontal for all points to the right of the kink, but undefined at the kink since we can draw infinitely many tangent lines at the kink, each with a different slope.

(f) $u(x,y) = Ax^{\alpha}y^{\beta}$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{\alpha A x^{\alpha-1}y^{\beta}}{\beta A x^{\alpha}y^{\beta-1}} = \frac{\alpha y^{\beta-(\beta-1)}}{\beta x^{\alpha-(\alpha-1)}} = \frac{\alpha y}{\beta x}$$

The marginal rate of substitution is diminishing. As this consumer consumes more of good x, additional units of good y becomes more valuable to them.

6. Eric's preferences for books, x, and computers, y, can be represented with the following Cobb-Douglas utility function $u(x,y) = x^3y^2$.

   (a) Find Eric's marginal utility for books, MU_x, and for computers, MU_y.

   - We calculate Eric's marginal utilities by differentiating with respect to each respective variable,

   $$MU_x = \frac{\partial u(x,y)}{\partial x} = 3x^2y^2 \qquad MU_y = \frac{\partial u(x,y)}{\partial y} = 2x^3y$$

   (b) Are his preferences monotonic (i.e., weakly increasing in both goods)?

   - Since both of Eric's marginal utilities are strictly positive, his preferences are monotonic.

   (c) For a given utility level, $\overline{u}$, solve the utility function for y to obtain Eric's indifference curve.

   - Setting Eric's utility at $\bar{u}$, we obtain

   $$x^3y^2 = \bar{u}$$

   Solving Eric's utility function for y, we find his indifference curve,

   $$y = \sqrt{\frac{\bar{u}}{x^3}}.$$

(d) Find Eric's marginal rate of substitution between x and y (MRS). Interpret your results.

- Eric's marginal rate of substitution can be found by taking the ratio of his marginal utility with respect to x to his marginal utility with respect to y,

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{3x^2y^2}{2x^3y} = \frac{3y}{2x}$$

Eric's marginal rate of substitution implies that for each additional unit of books he receives, he must be compensated with an additional $\frac{3}{2}$ units of computers in order to give up any further units of books.

(e) Are his preferences convex (i.e., bowed-in towards the origin)?

- The convexity of Eric's preferences can be found by differentiating his marginal rate of substitution with respect to x,

$$\frac{dMRS_{x,y}}{dx} = -\frac{3y}{2x^2} < 0$$

Since this is negative, Eric's marginal rate of substitution is decreasing in x, which implies that Eric's preferences are convex.

(f) Consider a given utility level of 10 utils. Plot his indifference curve in this case.

- Evaluating Eric's indifference curve $y = \sqrt{\frac{\bar{u}}{x^3}}$ from part (c) at a utility level of $\bar{u} = 10$, we find

$$y = \sqrt{\frac{10}{x^3}} = \frac{\sqrt{10}}{x^{3/2}}.$$

Plotting this indifference curve, we obtain figure 11.

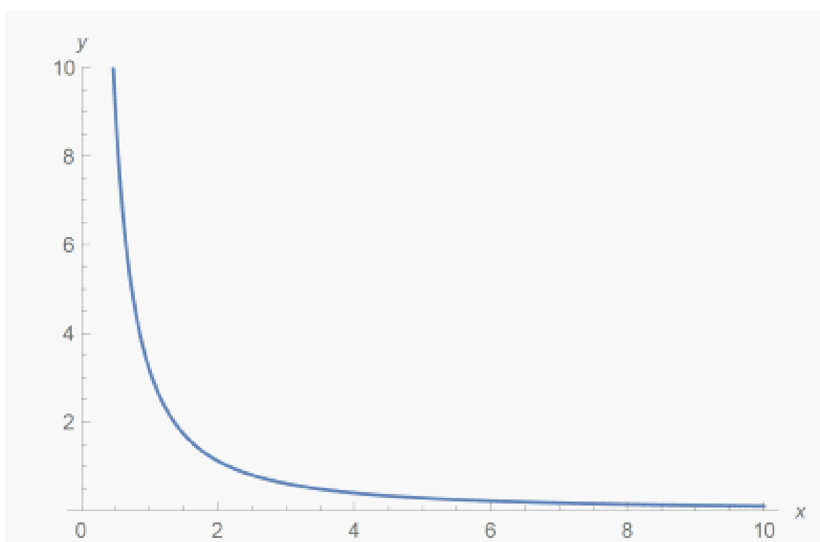

Figure 11. Cobb-Douglas.