EconS 501 Final Exam - December 8th, 2025

Show all your work clearly and make sure you justify all your answers.

NAME___________________________

1. A monopolist is considering to use price discrimination. The monopolist has identified two groups of consumers (type A and type B), their demand function is $p_A(q_A) = \alpha a - \frac{2b}{3}q_A$ and $p_B(q_B) = a - \frac{b}{3}q_B$, respectively. In addition the per unit cost is $c = 8$.

   (a) Find the equilibrium quantity and price. When does type A pay a higher price than type B? Discuss your results.

   - $$\begin{aligned} MR_A &= MC \\ \alpha a - \frac{4b}{3}q_A &= 8 \\ q_A &= \frac{3(\alpha a - 8)}{4b} \\ p_A &= \frac{\alpha a}{2} + 4 \end{aligned}$$

   - $$\begin{aligned} MR_B &= MC \\ a - \frac{2b}{3}q_B &= 8 \\ q_B &= \frac{3(a - 8)}{2b} \\ p_B &= \frac{a}{2} + 4 \end{aligned}$$

   - Hence, we have that $p_A > p_B$ if
   $$\begin{aligned} \frac{\alpha a}{2} + 4 &> \frac{a}{2} + 4 \\ \alpha &> 1 \end{aligned}$$

   - That is if the max. willingness to pay is higher for type A than type B.

   (b) Under which conditions does type A have a more inelastic demand than type B?

   - First we need to identify the direct demand for type A and B:
   $$\begin{aligned} q_A(p_A) &= \frac{3}{2b}\alpha a - \frac{3}{2b}p_A \\ q_B(p_B) &= \frac{3}{b}a - \frac{3}{b}p_B \end{aligned}$$

   - then calculate the price-elasticity of demand for each type
   $$\begin{aligned} \varepsilon_A &= \frac{\partial q_A(p_A)}{p_A} \times \frac{p_A}{q_A} \\ &= -\frac{3}{2b} \times \frac{\frac{\alpha a}{2} + 4}{\frac{3(\alpha a - 8)}{4b}} \\ &= -\frac{(\alpha a + 8)}{(\alpha a - 8)} \end{aligned}$$

- $$\begin{aligned}\varepsilon_B &= \frac{\partial q_B(p_B)}{p_B} \times \frac{p_B}{q_B} \\ &= -\frac{3}{b} \times \frac{\frac{a}{2}+4}{\frac{3(a-8)}{2b}} \\ &= -\frac{(a+8)}{(a-8)}\end{aligned}$$

- Hence, type A is more inelastic than type B if

$$\begin{aligned}-\frac{(a+8)}{(a-8)} &< -\frac{(\alpha a+8)}{(\alpha a-8)} \\ \alpha &> 1\end{aligned}$$

2. A utility function is additively separable if it has the form

$$u(x) = \sum_{k=1}^{L} u_k(x_k)$$

For instance, in the context of three goods, an additively separable function would be $u(x) = u_1(x_1) + u_2(x_2) + u_3(x_3)$, where function $u_k(x_k)$ can be linear or nonlinear in the units of good k, x_k.

(a) For the context of two goods, provide at least two examples of utility functions that are additively separable, and two examples of utility functions that are not.

- First, note that the utility of good k, $u_k(x_k)$, could be $u_k(x_k) = ax_k$, $u_k(x_k) = a \ln x_k$ or $u_k(x_k) = ax_k^2$ where $a > 0$; or more generally, functions of the form $u_k(x_k) = ax_k^{\beta}$, where $a, \beta \in \mathbb{R}$.
- *Additively separable utility functions.* Consider, for instance, goods that are regarded as substitutes, with utility function

$$u(x, y) = ax + by,$$

where $a, b \in \mathbb{R}$; or, more generally, utility functions such as

$$u(x, y) = ax^{\beta} + by^{\delta},$$

where $a, b, \beta, \delta \in \mathbb{R}$. Note that the last example includes quasi-linear utility functions of the form

$$u(x, y) = ax^{\beta} + by,$$

as a special case (when $\delta = 1$).
- *Not additively separable utility functions.* Consider, for example, the Cobb-Douglas utility function

$$u(x, y) = ax^{\alpha}y^{\beta},$$

where $a, b, \alpha, \beta \in \mathbb{R}$; the utility function representing goods that are regarded as complements,

$$u(x, y) = \min \{ax, by\},$$

where $a, b \in \mathbb{R}$; or the Stone-Geary utility function

$$u(x, y) = a (x - \overline{x})^{\alpha} (y - \overline{y})^{\beta},$$

where $a, \alpha, \beta \in \mathbb{R}$ and $\overline{x}, \overline{y} > 0$.

(b) Show that the marginal utility of good k is a function of the units of good k alone. Interpret.

- Differentiating the utility function $u(x)$ with respect to x_k, we obtain

$$MU_k = \frac{\sum_{l \neq k} u_l(x_l)}{\partial x_k} = \frac{\partial u_k(x_k)}{\partial x_k}$$

since all other components of the utility function, $\sum_{l \neq k} u_l(x_l)$, do not include x_k as arguments. Intuitively, when increasing the units of good k, the consumer only cares about the additional utility he obtains from this good, but ignores the number of units of other goods he consumes (that is, there is no interaction between the utilities of different goods, nor on their marginal utilities). Denoting the marginal utility of good k as $u_k'(x_k(p, w))$, we can more compactly express our above result as

$$MU_k = u_k'(x_k(p, w))$$

(c) Show that the Walrasian and Hicksian demand functions imply that all goods must be normal rather than inferior. (For simplicity, you can assume that the utility of every good is strictly concave for every good, differentiability, and interior solutions.)

- First, we know that the following tangency condition holds both in the UMP and in the EMP

$$MRS_{k,l} = \frac{MU_k}{MU_l} = \frac{u_k'(x_k(p, w))}{u_l'(x_l(p, w))} = \frac{p_k}{p_l}$$

(Recall that both the UMP and EMP have this tangency condition between the indifference curve and the budget line in common. However, the UMP inserts this result into its constraint, the budget line; whereas the EMP inserts the above result into its constraint, the utility level that the individual must reach.) Rearranging the above tangency condition, we obtain

$$u_k'(x_k(p, w)) = \frac{p_k}{p_l} u_l'(x_l(p, w)).$$

- Since we seek to show that no good can be inferior, we must consider a wealth change, leaving all prices unchanged. If wealth w increases, the demand for at least one good (say, good l) has to increase by Walras' law (otherwise, the individual would be buying fewer units of all goods, thus not exhausting his wealth). We seek to show that the demand for the remaining good k must also increase, thus implying that all goods are normal.

- To see this, first note that if the demand for good l increases, its marginal utility decreases. Graphically, a decrease in $u_l'(x_l(p,w))$ implies that the line representing $\frac{p_k}{p_l}u_l'(x_l(p,w))$ shifts downwards, yielding a new crossing point to the right-hand side of the initial crossing point depicted in the above figure. As a consequence, the consumer demands a larger amount of good k, i.e., $x_k(p,w)$ increases, ultimately implying that good k must be normal. Since our analysis applies to any good k, all goods must be normal.

3. Consider that your preference relation over three bundles, x_1, x_2, and x_3, satisfies

$$x_1 \succ x_2$$
$$x_2 \succ x_3$$
$$x_3 \succ x_1$$

(a) Show that you can be wiped out of your wealth w, where $w > 0$. (Hint: Begin with x_3.)

- First, let me begin with $x_2 \succ x_3$. I am willing to pay an amount α, where $\alpha > 0$, to exchange x_3 for x_2, since $x_2 \succ x_3$, ending up with wealth $w - \alpha$. The monetary amount $\alpha > 0$ can be as small as necessary to induce me to exchange x_3 for x_2.
- Second, since $x_1 \succ x_2$, I am willing to pay an amount β, where $\beta > 0$, to exchange x_2 for x_1, ending up with wealth $w-\alpha-\beta$. (The monetary amount β can coincide with α, $\beta = \alpha$, or differ, $\beta \neq \alpha$, without affecting our final result.)
- Third, since $x_3 \succ x_1$, I am willing to pay an amount γ, where $\gamma > 0$, to exchange x_1 for x_3, ending up with wealth $w - \alpha - \beta - \gamma$. (The monetary amount γ can coincide with α, β, or none of them, without affecting our final result.)
- Defining the sum of monetary amounts I paid so far, $y \equiv \alpha + \beta + \gamma$, I end up with my original bundle, x_3, but my wealth is reduced from w to $w - y$. Repeating the above three steps for n rounds, my wealth is wiped out, that is, we can find a number of rounds $\overline{n} \in \mathbb{N}$ such that

$$w - (\overline{n}+1)\,y < 0 < w - \overline{n}y$$

meaning that, at round $\overline{n}$, my wealth is still positive; but going through another round, $\overline{n}+1$, will leave me with a negative wealth.

(b) Consider an individual with a preference relation that violates rationality because his preferences are incomplete or intransitive. Discuss.

- A rational preference relation must be complete and transitive. Suppose that my preference relation is, instead:
  - incomplete, then there exists a pair of alternatives x and y in set X which I cannot compare, implying that my preference relation is not well defined.

– intransitive, then there exists a "money pump" as in part (a) that can eliminate all my wealth after finite rounds of exchanges.

Thus, preference relation must be complete and transitive in order to be rational.

4. Consider a setting with N individuals, where every individual i simultaneously and independently chooses her exploitation level $e_i \geq 0$. The marginal cost of effort is symmetric across individuals, $c > 0$. For compactness, denote by $e_{-i} = (e_1, ..., e_{i-1}, e_{i+1}, ...e_N)$ the profile of effort levels by i's rivals, E the sum of all individuals' efforts, and $E_{-i} = E - e_i$ the aggregate effort of all i's rivals. The utility function for individual i is given by

$$u_i(e_i, e_{-i}) = A(e_i + E_{-i})e_i - ce_i$$

In addition, assume that $A(\cdot)$ represents an outcome function which is strictly decreasing in aggregate effort E. This outcome function can represent several economic contexts, such as: (1) *a common-pool resource,* where $A(E) = \frac{f(E)}{E}$ indicates the average appropriation accruing to every player i, with $f(E)$ capturing total appropriation (e.g., total catches by all fishermen), as in Dasgupta and Heal (1979); (2) *Cournot competition*, where $A(E)$ represents the inverse demand function, which decreases in aggregate output, e.g., $A(E) = a - bE$; and (3) *rent-seeking contests* where $A(E) = \frac{e_i}{E}$ indicates the probability that player i wins the prize (e.g., promotion in a company), which is also decreasing in total effort.

(a) *Competitive equilibrium.* Find the implicit function that defines the equilibrium effort level that every individual i chooses. Define under which conditions the equilibrium effort you found is a utility maximum rather than a minimum.

- Every individual i solves

$$\max_{e_i} \quad A(e_i + E_{-i})e_i - ce_i$$

Taking first-order condition with respect to effort e_i, we obtain

$$A(E) + e_i A'(E) = c$$

In words, every individual i chooses an effort level e_i such that the marginal benefit from *individual* effort (left-hand side of the above equation) coincides with its own marginal cost from effort, c (right-hand side).

- Taking the second-order condition with respect to e_i, we obtain

$$2A'(E) + e_i A''(E)$$

Therefore, for the equilibrium effort to be a maximum, this second-order condition must be negative, i.e., $2A'(E) + e_i A''(E) < 0$. By definition, we know that the outcome of aggregate effort is decreasing, $A'(E) < 0$. If we assume that it is concave, $A''(E) < 0$, the second-order condition holds. Alternatively, function $A(E)$ can be convex as long as it is not "extremely convex", that is, $2A'(E) < e_i A''(E)$.

(b) *Social optimum.* Assume that a social planner considers the sum of every individual's utility as a measure of social welfare. Find the profile of effort levels that maximize social welfare (again, an implicit equation).

- The social planner's problem can be stated as

$$\max_{e_1, e_2, \ldots e_N} \sum_{i=1}^{N} u_i(e_i, e_{-i}) = \sum_{i=1}^{N} A(e_i + E_{-i})e_i - ce_i$$

  Taking the first-order conditions with respect to every e_i, we obtain

$$A(E) + A'(E)(e_1 + e_2 + ... + e_N) - c = 0$$

  where $E = \sum_{i=1}^{N} e_i$ denotes aggregate effort. Rearranging this expression, we obtain the profile of effort levels that maximize social welfare, which is the solution to the following implicit equation

$$A(E) + EA'(E) = c$$

- Intuitively, this equations says that the social planner chooses the profile of effort levels $e^* = (e_1, e_2, ...e_N)$ such that the marginal benefit of the *aggregate* effort (left-hand side of the equation) coincides with the marginal cost, c (right-hand side).

(c) *Comparison.* Compare your results from parts (a) and (b), showing that equilibrium effort is socially excessive. Interpret your results in terms of the above three economic contexts discussed above.

- In part (a), considering the welfare of each player, the equilibrium effort is determined by the equation

$$A(E) + e_i A'(E) - c = 0 \tag{1}$$

  In part (b), considering the sum of every individual's utility, the equilibrium effort is determined by the equation

$$A(E) + EA'(E) - c = 0 \tag{2}$$

  Comparing equations (1) and (2), since $A(\cdot)$ is strictly decreasing in E, we can see that players' effort in the competitive equilibrium is socially excessive. As a result, the aggregate socially optimal effort, E^{SO}, is lower than the competitive equilibrium effort, E^*, implying that, $A(E^{SO}) > A(E^*)$. The high quantity of aggregate effort generated by the competitive equilibrium results in a welfare loss due to the presence of negative externalities.

- *Common-pool resource interpretation.* When outcome function $A(E) = \frac{f(E)}{E}$ represents the average appropriation accruing to every player i (e.g., tons of fish captured by fisherman i), the payoff function for each individual i can be written as

$$u_i(e_i, e_{-i}) = \frac{e_i}{E} f(E) - ce_i \tag{3}$$

- *Rent-seeking interpretation.* Alternatively, equation (3) can be interpreted as the payoff function in a rent-seeking contest with $\frac{e_i}{E}$ as the probability that player i wins the prize. Intuitively, individual i's probability of winning increases in his own effort e_i, but decreases in the effort other players exert, as captured by aggregate effort E. A higher aggregate effort implies a lower probability of winning the prize. Hence, given that $E^* > E^{SO}$, the probability of winning the contest is higher when maximizing the social welfare than when maximizing individual welfare.
- *Cournot competition interpretation.* When outcome function $A(E)$ represents the inverse demand function, the inefficiency of the noncooperative equilibrium relative to the social optimum is equivalent to showing that collusion leads to lower aggregate production ($E^{SO} < E^*$) but higher prices ($A(E^{SO}) > A(E^*)$). Hence, collusive equilibrium results in higher profits.