EconS 501 Midterm #2 - November 18th, 2025

Show all your work clearly and make sure you justify all your answers.
NAME__________________________

1. CConsider an industry composed by two firms that recycle used products. Firm i's marginal cost from recycling is c_i while firm j's marginal cost is c_j, where $c_i > c_j > 0$. Each firm receives a benefit from recycling equal to

$$B_i(r_i, r_j) = 1 - r_i + \beta_i r_j,$$

where r_i denotes firm i's recycling output, r_j indicates firm j's recycling output. In addition, $1 > c_i, c_j$, $\beta_i \in [0,1]$ and $\beta_i \neq \beta_j$. Intuitively, parameter β_i measures how much firm j's recycling decisions benefits firm i (a positive externality). That is, when $\beta_i = 1$ firm i fully benefits from every unit of firm j's recycling activity, whereas when $\beta_i = 0$ firm j's recycling does not produce any benefit on firm i. Firm j's benefit from recycling is symmetric, that is, $B_j(r_i, r_j) = 1 - r_j + \beta_j r_i$.

(a) Set up every firm i's profit-maximization problem, find its best response function $r_i(r_j)$, and discuss whether firms recycling activities are strategic substitutes or complements.

- Every firm i solves

$$\max_{r_i \geq 0} \ \pi_i(r_i, r_j) = (1 - r_i + \beta_i r_j) r_i - c_i r_i$$

Differentiating with respect to its recycling output r_i, and solving for r_i, we find a best response function

$$r_i(r_j) = \frac{1 - c_i}{2} + \frac{\beta_i}{2} r_j$$

Graphically, this best response function originates at $\frac{1-c_i}{2}$ and increases in firm j's recycling r_j at a rate of $\frac{\beta_i}{2}$, indicating that firms' recycling activities are strategic complements. A symmetric argument for firm j yields best response function

$$r_j(r_i) = \frac{1 - c_j}{2} + \frac{\beta_j}{2} r_i$$

which is also increasing in its rival's recycling activities, r_i.

(b) Identify the equilibrium level of recycling every firm selects, i.e., r_i^* and r_j^*, respectively. How are your equilibrium results affected by costs c_i and c_j? How are they affected by parameters β_i and β_j?

- Simultaneously solving the two best response functions yields equilibrium recycling levels of

$$r_i^* = \frac{2(1 - c_i) + \beta_i(1 - c_j)}{4 - \beta_i \beta_j} \quad \text{and} \quad r_j^* = \frac{2(1 - c_j) + \beta_j(1 - c_i)}{4 - \beta_i \beta_j}$$

Note that firm i's equilibrium recycling r_i^* is positive as long as $c_i < 1 + \frac{\beta_i(1-c_j)}{2}$. As a curiosity, note that when positive externalities are absent, $\beta_i = \beta_j = 0$, this condition on c_i simplifies to $c_i < a$ as in similar simultaneous-move games without externalities; whereas when positive externalities are maximal, $\beta_i = \beta_j = 1$, the condition on c_i becomes $c_i < a + \frac{(a-c_j)}{2}$. Since the right-hand side of this condition on c_i increases in β_i, firm i is more willing to choose positive recycling amounts as positive externalities it receives from firm j become more substantial.

- *Comparative statics - Costs.* Note that

$$\begin{aligned} \frac{\partial r_i^*}{\partial c_i} &= -\frac{2}{4-\beta_i\beta_j} < 0 \text{ and} \\ \frac{\partial r_i^*}{\partial c_j} &= -\frac{\beta_j}{4-\beta_i\beta_j} < 0, \end{aligned}$$

indicating that if firm i or its competitor become less efficient (cost c_i or c_j increases) firm i responds reducing its equilibrium recycling.

- *Comparative statics - Positive externality.* In addition,

$$\begin{aligned} \frac{\partial r_i^*}{\partial \beta_i} &= \frac{2\beta_j(1-c_i)+4(1-c_j)}{\left(4-\beta_i\beta_j\right)^2} > 0, \text{ and} \\ \frac{\partial r_i^*}{\partial \beta_j} &= \frac{\beta_i\left[2(1-c_i)+4(1-c_j)\right]}{\left(4-\beta_i\beta_j\right)^2} > 0 \end{aligned}$$

suggesting that if the positive externality becomes stronger firms respond increasing their recycling. However, firm i responds more significantly to an increase in the positive externality it receives from its rival, β_i, than when its rival receives a larger externality from firm i, β_j.

(c) *Social optimum.* Assume that social welfare only considers the sum of every firms' profits. Identify the socially optimal levels of recycling, i.e., r_i^{SO} and r_j^{SO}.

- A social planner considering both firms' profits (or a partnership between both firms that considers both of their profits) selects the recycling level for both firms, (r_i, r_j), that solves

$$\begin{aligned} &\max_{r_i, r_j} \pi_i(r_i, r_j) + \pi_j(r_i, r_j) \\ &= \left[(1 - r_i + \beta_i r_j) r_i - c_i r_i\right] + \left[\left(1 - r_j + \beta_j r_i\right) r_j - c_j r_j\right] \end{aligned}$$

Taking first-order conditions with respect to r_i and r_j yields, respectively,

$$\begin{aligned} 1 - 2r_i + \beta_i r_j - c_i + \beta_j r_j &= 0, \text{ and} \\ 1 - 2r_j + \beta_j r_i - c_j + \beta_i r_i &= 0. \end{aligned}$$

Simultaneously solving for r_i and r_j in the above two first-order conditions yields the socially optimal levels of recycling

$$r_i^{SO} = \frac{2(a-c_i) + \left(\beta_i + \beta_j\right)(a - c_j)}{\left(4 - \beta_i\beta_j\right)\left(2 + \beta_i + \beta_j\right)}$$

for every firm i.

(d) *Comparison.* Compare the equilibrium recycling amounts that you found in part (b) and at the social optimum (from part c). Interpret your findings.

- Comparing r_i^* and r_i^{SO}, we find that $r_i^* < r_i^{SO}$ since

$$\frac{2(1-c_i)+\beta_i(1-c_j)}{4-\beta_i\beta_j} < \frac{2(1-c_i)+\left(\beta_i+\beta_j\right)(1-c_j)}{\left(4-\beta_i\beta_j\right)\left(2+\beta_i+\beta_j\right)}$$

simplifies to

$$0 < \frac{\beta_j(1-c_j)}{2+\beta_i+\beta_j}.$$

Intuitively, when every firm independently chooses its own recycling amount, it ignores the positive externality that this recycling produces on its rival. However, the social planner internalizes this externality, thus recommending a larger amount of recycling.

2. Consider an agent who exerts an effort level e, where $e \geq 0$, to generate output y that is subject to output shocks ε (e.g., weather conditions affecting the quality of harvest, machine breakdown causing product failure, etc.). Output then behaves as follows

$$y = ge + \varepsilon$$

where g denotes the agent's output efficiency, and shock ε follows a normal distribution, $N\left(0,\sigma^2\right)$, with mean zero and variance $\sigma^2 > 0$. The agent earns a wage of $w = sy$, where $0 \leq s \leq 1$ represents his output shares (e.g., commission from sales of the products). In addition, the agent incurs a cost to exert effort e, given by

$$c(e) = \frac{1}{2}e^2$$

which is increasing and convex in his effort level e.

The agent's payoff comes from the utility of earning wage w minus his cost of effort, where

$$U = u(w) - c(e)$$

Specifically, his utility function follows the negative exponential form of

$$u(w) = 1 - \exp(-\eta w) \tag{1}$$

(a) What is the agent's Arrow-Pratt coefficient of absolute risk aversion, $r_A(w)$. How does it vary with his wage?

- The absolute risk aversion parameter is defined by $r_A(w) \equiv -\frac{u''(w)}{u'(w)}$, where

$$u'(w) \equiv \frac{\partial u(w)}{\partial w} = \eta \exp(-\eta w)$$
$$u''(w) \equiv \frac{\partial u^2(w)}{\partial w^2} = -\eta^2 \exp(-\eta w)$$

Therefore, the absolute risk aversion parameter is

$$r_A(w) \equiv -\frac{u''(w)}{u'(w)} = -\frac{-\eta^2 \exp(-\eta' w)}{\eta \exp(-\eta' w)} = \eta$$

which is independent of his wage.

(b) Find the certainty equivalent of the agent.

- *Method 1.* The agent's expected utility from earning wage w is

$$\begin{aligned} E[u(w)] &= E[1 - \exp(-\eta w)] \\ &= 1 - \int_{-\infty}^{+\infty} \exp(-\eta w) f(w)\, dw \end{aligned}$$

Recall that the agent's wage follows a normal distribution, with density

$$f(w) = \frac{1}{\sqrt{2\pi}\sigma_w} \exp\left(-\frac{1}{2}\left(\frac{w-\mu_w}{\sigma_w}\right)^2\right)$$

where μ_w denotes the expected wage, while σ_w represents its standard deviation. We can then rewrite the agent's expected utility as

$$\begin{aligned} E[u(w)] &= 1 - \int_{-\infty}^{+\infty} \exp(-\eta w) \frac{1}{\sqrt{2\pi}\sigma_w} \exp\left(-\frac{1}{2}\left(\frac{w-\mu_w}{\sigma_w}\right)^2\right) dw \\ &= 1 - \frac{1}{\sqrt{2\pi}\sigma_w} \int_{-\infty}^{+\infty} \exp \underbrace{\left(-\eta w - \frac{1}{2}\left(\frac{w-\mu_w}{\sigma_w}\right)^2\right)}_{\text{Term } A} dw \qquad (2) \end{aligned}$$

Consider term A inside the exponent,

$$\begin{aligned} -\eta w - \frac{1}{2}\left(\frac{w-\mu_w}{\sigma_w}\right)^2 &= -\frac{w^2 - 2\mu_w w + 2\eta\sigma_w^2 w + \mu_w^2}{2\sigma_w^2} \\ &= -\frac{w^2 - 2(\mu_w - \eta\sigma_w^2) w + \mu_w^2}{2\sigma_w^2} \\ &= -\frac{[w - (\mu_w - \eta\sigma_w^2)]^2 + (2\eta\mu_w\sigma_w^2 - \eta^2\sigma_w^4)}{2\sigma_w^2} \\ &= -\frac{[w - (\mu_w - \eta\sigma_w^2)]^2}{2\sigma_w^2} - \left(\eta\mu_w - \frac{\eta^2\sigma_w^2}{2}\right) \end{aligned}$$

Substituting the above result into expression (2), we find that the agent's expected utility from working for the firm can be expressed as

$$\begin{aligned} E_A[u(w)] &= 1 - \frac{1}{\sqrt{2\pi}\sigma_w} \int_{-\infty}^{+\infty} \exp\left(-\frac{[w-(\mu_w - \eta\sigma_w^2)]^2}{2\sigma_w^2} - \left(\eta\mu_w - \frac{\eta^2\sigma_w^2}{2}\right)\right) dw \\ &= 1 - \exp\left(-\left(\eta\mu_w - \frac{\eta^2\sigma_w^2}{2}\right)\right) \underbrace{\frac{1}{\sqrt{2\pi}\sigma_w} \int_{-\infty}^{+\infty} \exp\left(-\frac{[w-(\mu_w - \eta\sigma_w^2)]^2}{2\sigma_w^2}\right) dw}_{=1 \text{ for integrating the PDF of } N(\mu_w - \eta\sigma_w^2, \sigma_w^2)} \\ &= 1 - \exp\left(-\eta\left(\mu_w - \frac{1}{2}\eta\sigma_w^2\right)\right) \qquad (3) \end{aligned}$$

Since the agent's utility function is a one-to-one mapping from wage to utility, we can define an inverse function, $u^{-1}(u)$, that maps the agent's utility to his certainty equivalent payment, as follows.

$$\begin{aligned}\exp(-\eta w) &= 1 - u(w) \\ -\eta w &= \log(1 - u(w)) \\ w &= -\frac{1}{\eta}\log(1 - u(w))\end{aligned}$$

so that the certainty equivalent wage is the payment that makes the agent indifferent to accepting the expected utility, and is given by

$$\begin{aligned}CE\left(\mu_w, \sigma_w^2\right) &= -\frac{1}{\eta}\log\left(1 - E_A\left[u(w)\right]\right) \\ &= -\frac{1}{\eta}\log\left(1 - \left[1 - \exp\left(-\eta\left(\mu_w - \frac{1}{2}\eta\sigma_w^2\right)\right)\right]\right) \\ &= -\frac{1}{\eta}\log\left(\exp\left(-\eta\left(\mu_w - \frac{1}{2}\eta\sigma_w^2\right)\right)\right) \\ &= -\frac{1}{\eta}\left(-\eta\left(\mu_w - \frac{1}{2}\eta\sigma_w^2\right)\right) \\ &= \mu_w - \frac{1}{2}\eta\sigma_w^2\end{aligned}$$

where his certainty equivalent wage, $CE(\mu_w, \sigma_w^2)$, increases linearly in the expected wage, μ_w, and decreases in the variance of output shocks σ_w^2 and risk aversion η. Specifically, his expected wage and variance are

$$\begin{aligned}\mu_w &= E[w] = E\left[s\underbrace{(ge + \varepsilon)}_{y}\right] = sge + s\underbrace{E[\varepsilon]}_{0} = sge \\ \sigma_w^2 &= Var(w) = Var(s(ge + \varepsilon)) = Var(s\varepsilon) = s^2\sigma^2\end{aligned}$$

such that his certainty equivalent wage can be equivalently expressed as

$$CE(e) = sge - \frac{1}{2}\eta s^2\sigma^2$$

Intuitively, the agent would accept a lower certainty equivalent wage when output shocks, σ, becomes more volatile or when he becomes more risk averse (higher η). In contrast, he demands a higher certainty equivalent when he exerts a larger effort level e and when effort becomes more productive, that is, a higher g.

- *Method 2.* Define $CE(w)$ and $RP(w)$ to be the certainty equivalent and risk premium of the agent, where $CE(w) \equiv E(w) - RP(w)$ and $w \sim N(sge, s^2\sigma^2)$. Since by definition the agent is indifferent between accepting the certainty equivalent and stochastic wage, specifically, $u(w - RP(w)) =$

$E\left[u\left(w+s\varepsilon\right)\right]$; we take the first (second) order Taylor series expansion on the left (right) side that yields

$$\begin{aligned} u\left(w\right)-u'\left(w\right)\cdot RP\left(w\right) &= E\left[u\left(w\right)+u'\left(w\right)\cdot s\varepsilon+\frac{1}{2}u''\left(w\right)\cdot s^{2}\varepsilon^{2}\right] \\ &= \underbrace{E\left[u\left(w\right)\right]}_{=u(w)}+su'\left(w\right)\underbrace{E\left[\varepsilon\right]}_{=0}+\frac{s^{2}}{2}u''\left(w\right)\underbrace{E\left[\varepsilon^{2}\right]}_{=\sigma^{2}} \\ &= u\left(w\right)+\frac{s^{2}\sigma^{2}}{2}u''\left(w\right) \end{aligned}$$

Cancelling out $u\left(w\right)$ on both sides, we obtain

$$RP\left(w\right)=-\frac{s^{2}\sigma^{2}}{2}\cdot\frac{u''\left(w\right)}{u'\left(w\right)}=\frac{s^{2}\sigma^{2}}{2}r_{A}\left(w\right)$$

Therefore, certain equivalent of the agent becomes

$$\begin{aligned} CE\left(w\right) &\equiv E\left(w\right)-RP\left(w\right) \\ &= E\left(w\right)-\frac{s^{2}\sigma^{2}}{2}r_{A}\left(w\right) \\ &= sge-\frac{1}{2}\eta s^{2}\sigma^{2} \end{aligned}$$

(c) Find the agent's optimal effort e^{*}. How does it vary with his output share s, output efficiency g, risk aversion η, and output shocks σ?

- The agent's expected payoff now becomes

$$\begin{aligned} E\left[U\left(e\right)\right] &= E\left[u\left(w\right)\right]-E\left[c\left(e\right)\right] \\ &= CE\left(e\right)-c\left(e\right) \\ &= \left(sge-\eta\frac{s^{2}\sigma^{2}}{2}\right)-\frac{1}{2}e^{2} \end{aligned}$$

  where the second line emanates from the certainty equivalent payment, $CE\left(e\right)$, that makes the agent indifferent to accepting the risky outcome $u\left(w\right)$. Differentiating his expected payoff with respect to effort e, and assuming interior solutions, we find

$$\frac{dE\left[U\left(e\right)\right]}{de}=sg-e=0$$

  such that the optimal effort becomes

$$e^{*}=sg.$$

- The optimal effort e^{*} is increasing in his output share s and in output efficiency g, but is independent of risk aversion (parameter η) and the volatility of output shocks σ^{2}. Intuitively, when the agent obtains a higher share of the output, or when he becomes more efficient at producing the output, he exerts more effort to generate a higher output. Since in expectation his share of output is not affected by risk aversion or output shocks, his optimal effort is not affected by parameters η or σ^{2}.

3. Consider two consumers with utility functions over two goods, x_1 and x_2, given by

$$\begin{aligned} u_A &= \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) \quad \text{for consumer } A\text{, and} \\ u_B &= \log(x_1^B) + x_2^B - \frac{1}{2}\log(x_1^A) \quad \text{for consumer } B. \end{aligned}$$

where the consumption of good 1 by individual $i = \{A, B\}$ creates a negative externality on individual $j \neq i$ (see the third term, which enters negatively on each individual's utility function). For simplicity, consider that both individuals have the same wealth, m, and that the price for both goods is 1.

(a) *Unregulated equilibrium.* Set up consumer A's utility maximization problem, and determine his demand for goods 1 and 2, as x_1^A and x_2^A. Then operate similarly to find consumer B's demand for good 1 and 2, as x_1^B and x_2^B.

- Consumer A chooses x_1^A and x_2^A to solve

$$\max_{(x_1^A, x_2^A)} \quad \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$

$$\text{subject to} \quad x_1^A + x_2^A = M$$

The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M - x_1^A - x_2^A),$$

which yields first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - x_1^A - x_2^A = 0$$

Solving for x_1^A, we obtain $\frac{1}{x_1^A} = 1$, i.e., $x_1^A = 1$, which implies $M - 1 - x_2^A = 0$, or $x_2^A = M - 1$. Hence, consumer A's optimal consumption is

$$x_1^A = 1 \quad \text{and} \quad x_2^A = M - 1$$

A similar argument applies to consumer B,

$$x_1^B = 1 \quad \text{and} \quad x_2^B = M - 1$$

(b) *Social optimum.* Calculate the socially optimal amounts of x_1^A, x_2^A, x_1^B and x_2^B, considering that the social planner maximizes a utilitarian social welfare function, namely, $W = U_A + U_B$.

- The socially optimal consumption in this case solves

$$\max_{(x_1^A, x_2^A)} U^A + U^B \quad \text{subject to } x_1^A + x_2^A = M \text{ and } x_1^B + x_2^B = M$$

The Lagrangian for this social planner's problem is

$$\mathcal{L} = \frac{1}{2}\log(x_1^A) + \frac{1}{2}\log(x_1^B) + x_2^A + x_2^B + \lambda^A(M - x_1^A - x_2^A) + \lambda^B(M - x_1^B - x_2^B)$$

Taking first-order conditions, we find the socially optimal consumption profile:

$$x_1^A = \frac{1}{2} \text{ and } x_2^A = M - \frac{1}{2}$$

$$x_1^B = \frac{1}{2} \text{ and } x_2^B = M - \frac{1}{2}$$

Intuitively, the social planner recommends a lower consumption of good 1 (the good that generates the negative externality), and an increase in the consumption of good 2, for both individuals.

(c) *Restoring efficiency.* Show that the social optimum you found in part (b) can be induced by a tax on good 1 (so the after-tax price becomes $1+t$) with the revenue returned equally to both consumers in a lump-sum transfer.

- With tax t^A placed on good 1 and with lump-sum transfer T^A, consumer A solves

$$\max_{(x_1^A, x_2^A)} \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$

$$\text{subject to } (1 + t^A)x_1^A + x_2^A = M + T^A$$

where note that the price of good 1 increased from 1 to $(1 + t^A)$, but this consumer also sees his wealth increase by the lump sum T^A. The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M + T^A - (1 + t^A)x_1^A - x_2^A)$$

Taking first-order conditions, we obtain

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A(1 + t^A) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M + T^A - (1 + t^A)x_1^A - x_2^A = 0$$

Simultaneously solving for x_1^A and x_2^A, we find that consumer A's consumption bundles after introducing the tax become

$$x_1^A = \frac{1}{1 + t^A} \text{ and } x_2^A = M + T^A - 1$$

Similarly we find the optimal consumption of consumer B who pays tax t^B on good 1 and receives T^B as a lump-sum transfer:

$$x_1^B = \frac{1}{1+t^B} \text{ and } x_2^B = M + T^B - 1$$

- *Comparison.* Comparing the optimal consumption levels found in part (b) with the equilibrium outcomes found in part (c), the tax imposed on any individual $i = A, B$ must hence satisfy

  $$\frac{1}{2} = \frac{1}{1+t^i},$$

  which would guarantee that equilibrium and socially optimal amounts coincide. Solving for the tax t^i yields $t^i = \$1$. Hence, by setting a tax of $t^i = \$1$ on the consumption of good 1, and returning the tax revenue to this individual in a lump-sum transfer, efficiency is restored, yielding a consumption

  $$x_1^i = \frac{1}{1+1} = \frac{1}{2} \text{ of good 1,}$$

  and

  $$\begin{aligned} x_2^i &= M + T^i - 1 \\ &= M + \frac{1}{2} - 1 = M - \frac{1}{2} \text{ of good 2,} \end{aligned}$$

  as described in the socially optimal amounts found in part (b).

4. Consider a context in which the social planner cannot observe firm's efficiency level θ, where $\theta \in [0, 1]$. A value of θ close to zero (one) indicates a very efficient (inefficient, respectively) firm. Firms generate a negative externality (pollution) for each unit of output x. Two policies are available to address this negative externality: (i) taxes and (ii) quotas. What policy should the social planner use when the marginal damage function is not very sensitive to pollution? [Provide reasons and details supporting your answer.]

- *Slides 105-110 - Chapter 9*