Homework #7 - Due on November 13th

1. Let us consider a market with two firms, Firm A and Firm B, producing a homogeneous good. However, Firm A generates more pollution than Firm B during the production process as explained below. Firm i's marginal production costs are given by c_i where $i = \{A, B\}$, where c_B is strictly higher than c_A. In addition, the social welfare function that the regulator uses to set emission fees on these firms is

$$SW = CS + PS + T - Env$$

where CS is the consumer surplus, PS is the producer surplus, $T = t\,(q_A + q_B)$ is the tax revenue from emission fees on both firms, and $Env = d_A(q_A)^2 + d_B(q_B)^2$ is the environmental damage from the production of both goods, where $d_A \geq d_B$. Finally, the inverse demand function of firm $i = \{A, B\}$ is

$$p_i(q_i, q_j) = 1 - q_i - q_j \quad \text{where } j = \{A, B\} \text{ and } j \neq i.$$

where q_i denotes output.

a. *No regulation.* Find equilibrium output levels when firms do not face emission fees. Interpret. [**10 Points**]

- Every firm i maximizes its profits as follows

$$\max_{q_i \geq 0} \; p_i(q_i, q_j)q_i - c_i q_i = (1 - q_i - q_j)\, q_i - c_i q_i$$

Differentiating with respect to q_i yields

$$1 - 2q_i - q_j - c_i = 0$$

Solving for q_i, we obtain firm i's best response function

$$q_i(q_j) = \frac{1 - c_i}{2} - \frac{1}{2} q_j$$

which originates at $\frac{1-c_i}{2}$, and decreases in q_j at a rate of $\frac{1}{2}$.

- Firm j's best response function is symmetric. We can then insert $q_j(q_i)$ into $q_i(q_j)$, as follows

$$q_i = \frac{1 - c_i}{2} - \frac{1}{2}\overbrace{\left(\frac{1 - c_j}{2} - \frac{1}{2}q_i\right)}^{q_j(q_i)}$$

Solving for q_i, we obtain the equilibrium output under no regulation

$$q_i^* = \frac{2(1-c_i)-(1-c_j)}{3}.$$

Since firms face different marginal production cost, $c_B > c_A$, equilibrium output becomes $q_A^* = \frac{1-2c_A+c_B}{3}$ and $q_B^* = \frac{1-2c_B+c_A}{3}$. In this case firm A produces more output than firm B since $c_B > c_A$.

b. *Regulation.* Find equilibrium output levels when firms face any emission fee t. Interpret. **[10 Points]**

When the firm is subject to an emission fee t per unit of output, the above maximization problem becomes

$$\max_{q_i \geq 0} \; p_i(q_i, q_j)\, q_i - c_i q_i - t q_i = (1 - q_i - q_j)\, q_i - (c_i + t)\, q_i$$

where only the cost in the last term changed, from c_i to $(c_i + t)$. Following the same steps as in part (a), we find equilibrium output

$$q_i^* = \frac{2\left[1-(c_i+t)\right]-\left[1-(c_j+t)\right]}{3}.$$

which coincides with the expression found in part (b), except for the fact that each firm's marginal production cost is increased the emission fee.
Since firms face different marginal cost and fee, $c_B > c_A$ and t, equilibrium output simplifies to $q_A^* = \frac{1-2c_A+c_B-t}{3}$ and $q_B^* = \frac{1-2c_B+c_A-t}{3}$. For each set of parameter values, every firm i sells fewer units when it is subject to the emission fee $t > 0$ than otherwise.

c. Identify the socially optimal output level for firm A, q_A^{SO}, and for firm B, q_B^{SO}. **[10 Points]**

The social planner chooses output levels q_A and q_B to maximize social welfare, as follows,

$$\begin{aligned}
&\max_{q_B, q_G \geq 0} \; CS + \pi_A + \pi_B - Env \\
&\frac{1}{2}(q_A^2 + 2q_A q_B + q_B^2) + (1 - q_A - q_B)q_A - c_A q_A \\
&+(1 - q_A - q_B)q_B - c_B q_B - d_A q_A^2 - d_B q_A^2
\end{aligned}$$

Differentiating with respect to q_A we obtain

$$q_A + q_B + 1 - 2q_A - q_B - c_A - q_B - 2d_A q_A = 0$$

Further simplifying, yields

$$(1+2d_A)\,q_A = 1 - q_B - c_A$$

Thus, solving for q_A, we find

$$q_A(q_B) = \frac{1-c_A}{1+2d_A} - \frac{1}{1+2d_A}q_B.$$

Differentiating with respect to q_B in the above maximization problem, we obtain a symmetric expression, $q_B(q_A)$. Inserting $q_B(q_A)$ into $q_A(q_B)$, and solving for q_A, yields the socially optimal output for the brown firm

$$q_A^{SO} = \frac{(c_B - c_A) + 2d_B\,(1-c_A)}{2(d_B + d_A\,(1+2d_B))}$$

By symmetry, the socially optimal output for the green firm is

$$q_B^{SO} = \frac{(c_A - c_B) + 2d_A\,(1-c_B)}{2(d_B + d_A\,(1+2d_B))}$$

d. Find the socially optimal fees (t) that induce firms to produce the socially optimal output levels found in part (c). Assume that $d_A = 2$ and $d_B = 1$,and $c_B = \frac{1}{4}$ and $c_A = 0$. **[20 Points]**

- The social optimal tax induces each type of firm to produce the socially optimal output q_i^{SO}. That is to say, we need to set $q_A(t) + q_B(t) = q_A^{SO} + q_B^{SO}$.Solving for t we obtain

$$\frac{1-2c_A+c_B-t}{3} + \frac{1-2c_B+c_A-t}{3} =$$

$$\frac{(1+2d_B)\,(1-c_A) - (1-c_B)}{(1+2d_A)\,(1+2d_B) - 1} + \frac{(1+2d_A)\,(1-c_B) - (1-c_A)}{(1+2d_A)\,(1+2d_B) - 1}$$

$$t = 0.34$$

which is always positive.

e. Now consider the case in which both firms invest in clean technology at a cost $F > 0$. This investment reduces the amount of pollution of firm A in αq_A, where $\alpha \in (0,1)$ and that of firm B in βq_A, where $\beta \in (0,1)$ and $\alpha \neq \beta$. That is, when α (or β) approaches to zero the firm is able to reduce almost all its emissions. Identify the the socially optimal fees (t) that induce firms to produce the socially optimal output levels. Represent it in general terms and then assume that $d_A = 2$ and $d_B = 1$,and $c_B = \frac{1}{4}$ and

$c_A = 0$. Compare your results with the socially optimal fees in part (d) and discuss under which conditions is this fee in part (d) higher than that in part (e). **[50 Points]**

- When the firm A is subject to an emission fee t per unit of output, the above maximization problem becomes

$$\max_{q_A \geq 0} \; (1 - q_A - q_B)\, q_A - c_A q_A - t\alpha q_A - F$$

and when firm B is subject to an emission fee t per unit of output, the above maximization problem becomes

$$\max_{q_B \geq 0} \; (1 - q_A - q_B)\, q_B - c_B q_B - t\beta q_A - F$$

F.O.C with respect to q_A and q_B yields

$$\begin{aligned} 1 - 2q_A - q_B - c_A - t\alpha &= 0 \\ 1 - 2q_B - q_A - c_B - t\beta &= 0 \end{aligned}$$

solving for q_A and q_B we obtain the BRF

$$\begin{aligned} q_A &= \frac{1 - c_A - t\alpha}{2} - \frac{q_B}{2} \\ q_B &= \frac{1 - c_B - t\beta}{2} - \frac{q_A}{2} \end{aligned}$$

Since firms face different marginal cost and fee, $c_B > c_A$ and t, equilibrium output simplifies to $q_A^* = \frac{1-2c_A+c_B-2t\alpha+t\beta}{3}$ and $q_B^* = \frac{1-2c_B+c_A-2t\beta+t\alpha}{3}$.
The social planner chooses output levels q_A and q_B to maximize social welfare, as follows,

$$\begin{aligned} &\max_{q_B, q_G \geq 0} \; CS + \pi_A + \pi_B - Env \\ &\frac{1}{2}(q_A^2 + 2q_A q_B + q_B^2) + (1 - q_A - q_B)q_A - c_A q_A - F \\ &+(1 - q_A - q_B)q_B - c_B q_B - d_A(\alpha q_A)^2 - d_B(\beta q_B)^2 \end{aligned}$$

Differentiating with respect to q_A we obtain

$$q_A + q_B + 1 - 2q_A - q_B - c_A - q_B - 2d_A\alpha^2 q_A = 0$$

Further simplifying, yields

$$\left(1 + 2d_A\alpha^2\right) q_A = 1 - q_B - c_A$$

Thus, solving for q_A, we find

$$q_A(q_B) = \frac{1-c_A}{1+2d_A\alpha^2} - \frac{1}{1+2d_A\alpha^2}q_B.$$

Differentiating with respect to q_B in the above maximization problem, we obtain a symmetric expression, $q_B(q_A)$. Inserting $q_A(q_B)$ into $q_B(q_A)$, and solving for q_A, yields the socially optimal output for the brown firm

$$q_A^{SO} = \frac{(c_B - c_A) + 2d_B\beta\,(1-c_A)}{2(d_B\beta + d_A\alpha\,(1+2d_B\beta))}$$

By symmetry, the socially optimal output for the green firm is

$$q_B^{SO} = \frac{(c_A - c_B) + 2d_A\alpha\,(1-c_B)}{2(d_A\alpha + d_B\beta\,(1+2d_A\alpha))}$$

The social optimal tax induces each type of firm to produce the socially optimal output q_i^{SO}. That is to say, we need to set $q_A(t) + q_B(t) = q_A^{SO} + q_B^{SO}$.Solving for t we obtain

$$\frac{1-2c_A+c_B-2t\alpha+t\beta}{3} + \frac{1-2c_B+c_A-2t\beta+t\alpha}{3} =$$

$$= \frac{(c_B - c_A) + 2d_B\beta\,(1-c_A)}{2(d_B\beta + d_A\alpha\,(1+2d_B\beta))} + \frac{(c_A - c_B) + 2d_A\alpha\,(1-c_B)}{2(d_A\alpha + d_B\beta\,(1+2d_A\alpha))}$$

and solving for t we obtain

$$t = \frac{d_B\beta^2(2c_A - c_B - 1) + d_A\alpha^2(1 - c_A - 2c_B - 2d_B\beta^2(c_A + c_B - 2))}{(\alpha+\beta)(d_B\beta^2 + d_A\alpha^2(1+2d_B\beta^2))}$$

If we assume that $d_A = 2$ and $d_B = 1$,and $c_B = \frac{1}{4}$ and $c_A = 0$

$$t = \frac{4\alpha^2(7\beta^2-1) - 5\beta^2}{4(\alpha+\beta)(\beta^2+\alpha^2(2+4\beta^2))}$$

Finally, in order to compare the two emissions let's assume that the investment in clean technology by firm A completely eliminates emissions, i.e., $\alpha = 0$, hence, in this case $t = \frac{-5}{4\beta}$, which is negative by definition since $\beta \in (0,1)$. Therefore, the investment in clean technology induces a subsidy and the result in part (e) is lower than that in part (d) where there is no investment in clean technology.