Homework # 6 - [Due on November 6th, 2025]

1. Consider a perfectly competitive industry with N symmetric firms, each with cost function $c(q) = F + cq$, where $F, c > 0$. Assume that the inverse demand is given by $p(Q) = a - bQ$, where $a > c$, $b > 0$, and where Q denotes aggregate output.

   (a) *Short-run equilibrium.* If exit and entry is not possible in the industry (assuming N firms remain active), find the individual production level of each firm.

   - Each invidual firm i solves the PMP

   $$\max_{q_i \geq 0} \ (a - bQ)q_i - (F + cq_i) = \left(a - bq_i - b\sum_{j \neq i} q_j\right) q_i - (F + cq_i)$$

   Taking first-order conditions with respect to q_i yields

   $$a - 2bq_i - b\sum_{j \neq i} q_j - c = 0$$

   and applying symmetry in equilibrium outputs, i.e., $q_1 = q_2 = ... = q_N$, we obtain an individual equilibrium output

   $$q_i = \frac{a - c}{b(N + 1)}$$

   for every firm $i \in N$. Note that this result is a function of the number of active firms in the industry, N.

   - In this setting, the equilibrium market price is

   $$p^* = a - b\underbrace{\left(N \cdot \frac{a - c}{b(N + 1)}\right)}_{Q = N \cdot q_i} = \frac{a + Nc}{N + 1}$$

   (b) *Long-run equilibrium.* Consider now that firms have enough time to enter the industry (if economic profits can be made) or to exit (if they make losses by staying in the industry). Find the long-run equilibrium number of firms in this perfectly competitive market.

   - In a long-run equilibrium of a perfectly competitive market, we know that firms must be making no economic profits, $\pi = 0$, as otherwise firms would still have incentives to enter or exit the industry. Hence, we first need to find the equilibrium profits that every individual firm i earns by producing the equilibrium output q_i found in part (a). In particular, these profits are

$$\pi_i = (a - b\underbrace{Nq_i}_{Q})q_i - (F + cq_i) = \frac{(a-c)^2}{b(N+1)^2} - F$$

setting them equal to zero and solving for N yields the long-run equilibrium number of firms, $\lfloor N^* \rfloor = \frac{a-c}{\sqrt{bF}} - 1$, where $\lfloor N \rfloor$ indicates the highest integer smaller or equal to N. For instance, if $a = b = 1$, $c = \frac{1}{4}$, and $F = \frac{1}{16}$ N^* becomes $N^* = 2$.

- More generally, note that the expression we found for equilibrium profits, π_i, is monotonically decreasing in the number of firms, N, for all parameter values, that is

$$\frac{\partial \pi_i}{\partial N} = -2b(a-c)^2(N-1) < 0$$

thus implying that equilibrium profits becomes zero for a sufficiently large number of firms.

2. A tax is to be levied on a commodity bought and sold in a competitive market. Two possible forms of tax may be used: In one case, a *per unit* tax is levied, where an amount t is paid per unit bought or sold. In the other case, an *ad valorem* tax is levied, where the government collects a tax equal to τ times the amount the seller receives from the buyer. Assume that a partial equilibrium approach is valid.

(a) Show that, with a per unit tax, the ultimate cost of the good to consumers and the amounts purchased are independent of whether the consumers or the producers pay the tax. As a guidance, let us use the following steps:

1. *Consumers:* Let p^c be the competitive equilibrium price when the *consumer* pays the tax. Note that when the consumer pays the tax, he pays $p^c + t$ whereas the producer receives p^c. State the equality of the (generic) demand and supply functions in the equilibrium of this competitive market when the consumer pays the tax.

   - If the per unit tax t is levied on the consumer, then he pays $p+t$ for every unit of the good, and the demand at market price p becomes $x\left(p+t\right)$. The equilibrium market price p^c is determinded from equalizing demand and supply:

   $$x\left(p^c + t\right) = q\left(p^c\right).$$

2. *Producers:* Let p^p be the competitive equilibrium price when the *producer* pays the tax. Note that when the producer pays the tax, he receives $p^p - t$

whereas the consumer pays p^p. State the equality of the (generic) demand and supply functions in the equilibrium of this competitive market when the producer pays the tax.

- On the other hand, if the per unit tax t is levied on the producer, then he collects $p - t$ from every unit of the good sold, and the supply at market price p becomes $q(p-t)$. The equilibrium market price p^p is determined from equalizing demand and supply:

$$x(p^p) = q(p^p - t).$$

(b) Show that if an equilibrium price p solves your equality in part (a), then $p + t$ solves the equality in (b). Show that, as a consequence, equilibrium amounts are independent of whether consumers or producers pay the tax.

- It is easy to see that p solves the first equation if and only if $p + t$ solves the second one. Therefore, $p^p = p^c + t$, which is the ultimate cost of the good to consumers in both cases. The amount purchased in both cases is

$$x(p^p) = x(p^c + t).$$

(c) Show that the result in part (b) is not generally true with an ad valorem tax. In this case, which collection method leads to a higher cost to consumers? [*Hint:* Use the same steps as above, first for the consumer and then for the producer, but taking into account that now the tax increases the price to $(1+\tau)p$. Then, construct the excess demand function for the case of the consumer and the producer.]

- If the ad valorem tax τ is levied on the consumer, then he pays $(1+\tau)p$ for every unit of the good, and the demand at market price p becomes $x((1+\tau)p)$. The equilibrium market price p^c is determined from equalizing demand and supply:

$$x((1+\tau)p^c) = q(p^c).$$

On the other hand, if the ad valorem tax τ is levied on the producer, he receives $(1+\tau)p$ for every unit of the good sold, and the supply at market price p becomes $q((1-\tau)p)$. The equilibrium market price p^p is determined from equalizing demand and supply:

$$x(p^p) = q((1-\tau)p^p).$$

Consider the excess demand function for this case:

$$z(p) = x(p) - q((1-\tau)p) \tag{1}$$

Since the demand curve $x(\cdot)$ is non-increasing and the supply curve $q(\cdot)$ is non-decreasing, $z(p)$ must be non-increasing. From (1) we have

$$\begin{aligned} z((1+\tau)p^c) &= x((1+\tau)p^c) - q((1-\tau)[(1+\tau)p^c]) = \\ &= x((1+\tau)p^c) - q((1-\tau^2)p^c) \geq \\ &\geq x((1+\tau)p^c) - q(p^c) = 0, \end{aligned}$$

where the inequality takes into account that $q(\cdot)$ is non-decreasing.

- Therefore, $z((1+\tau)p^c) \geq 0$ and $z(p^p) = 0$. Since $z(\cdot)$ is non-increasing, this implies that $(1+\tau)p^c \leq p^p$. In words, levying the ad valorem tax on consumers leads to a lower cost on consumers than levying the same tax on producers. (In the same way, it can be shown that levying the ad valorem tax on consumers leads to a higher price for producers than levying the same tax on producers).

(d) Are there any special cases in which the collection method is irrelevant with an ad valorem tax? [*Hint:* Think about cases in which the tax introduces the same wedge on consumers and producers (inelasticity). Then prove your statement by using the above argument on excess demand functions.]

- If the supply function $q(\cdot)$ is strictly increasing, the argument can be strengthened to obtain the strict inequality: $(1+\tau)p^c < p^p$. On the other hand, when the supply is perfectly inelastic, i.e., $q(p) = \overline{q}$ =constant, then yield

$$x((1+\tau)p^c) = \overline{q} = x(p^p),$$

and therefore $p^p = (1+\tau)p^c$. Here both taxes result in the same cost to consumers. However, producers still bear a higher burden when the tax is levied directly on them:

$$(1-\tau)p^p = (1-\tau)(1+\tau)p^c < p^c.$$

these prices are depicted in the next figure, where $x(p)$ reflects the demand function with no taxes and $x((1-\tau)p)$ represents the demand function with the ad valorem tax. While the inelastic supply curve guarantees that sales are unaffected by the tax (remaining at $\overline{q}$ units), the price that the producer receives drops from p^p to $(1-\tau)p^p$. Therefore, the two taxes are still not fully equivalent.

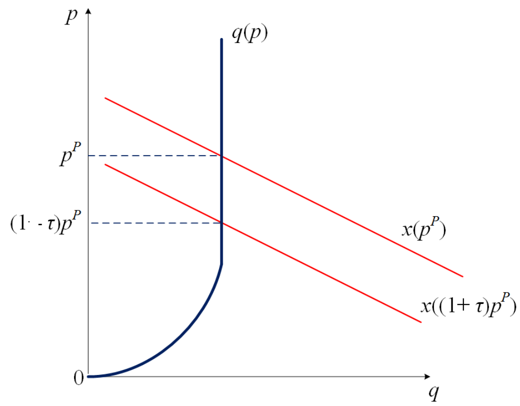

Figure 1. Introducing a tax.

- The intuition behind these results is simple: with a tax, there is always a wedge between the "consumer price"and the "producer price." Levying an ad valorem tax on the producer price, therefore, results in a higher tax burden (and a higher tax revenue) than levying the same percentage tax on consumers.

3. In our discussion of perfectly competitive markets, we considered that all firms produced a homogeneous good. However, our analysis can be easily extended to settings in which goods are heterogeneous. In particular, consider that every firm $i \in N$ faces a inverse demand function

$$p_i(q_i, q_{-i}) = \frac{\theta q_i^{\beta-1}}{\sum_{j=1}^{N} q_j^{\beta}}$$

where q_i denotes firm i's output, q_{-i} the output decisions of all other firms, i.e., $q_{-i} = (q_1, ..., q_{i-1}, q_{i+1}, ..., q_N)$, θ is a positive constant, and parameter $\beta \in (0, 1]$ captures the degree of substitutability. In addition, assume that every firm faces the same cost function $c(q_i) = F + cq_i$, where $F > 0$ denotes fixed costs and $c > 0$ represents marginal costs. Find the individual production level of every firm i, q_i^*, as a function of β. Interpret.

   - Every firm i's solves the following PMP

$$\max_{q_i} \frac{\theta q_i^{\beta-1}}{\sum_{j=1}^{N} q_j^{\beta}} q_i - (F + cq_i)$$

Taking first-order conditions with respect to q_i yields

$$\frac{\theta\left[\beta q_i^{\beta-1}\left(\sum_{j=1}^{N} q_j^{\beta}\right) - q_i^{\beta}\left(\beta q_i^{\beta-1}\right)\right]}{\left(\sum_{j=1}^{N} q_j^{\beta}\right)^2} - c = 0$$

In a symmetric equilibrium, output levels satisfy $q_i^* = q^*$ for every firm $i \in N$, thus simplifying the above expression to

$$\frac{\theta\beta q^{2\beta-1}(N-1)}{N^2 q^{2\beta}} - c = 0$$

Solving for q^* yields the individual equilibrium output

$$q^* = \frac{\theta\beta(N-1)}{N^2 c}$$

- *Comparative statics.* Differentiating q^* with respect to the substitutability parameter β we obtain

$$\frac{\partial q^*}{\partial \beta} = \frac{\theta(N-1)}{N^2 c} > 0$$

Hence, as goods become more differentiated (higher β), the equilibrium output level q^* rises. However, as more firms operate in this market, the increase in q^* becomes smaller since the derivative $\frac{\partial q^*}{\partial \beta}$ decreases in N, i.e.,

$$\frac{\partial\left(\frac{\partial q^*}{\partial \beta}\right)}{\partial N} = \frac{\theta N^2 c - \theta(N-1)2Nc}{(N^2 c)^2} < 0.$$