Recitation #7 - (10/17/2025)

1. State the independence axiom. Show that if indifference curves in the Machina triangle are *not* parallel straight lines, then the independence axiom is violated.

   - *Independence axiom.* A preference relation $\succsim$ over the space of simple lotteries $\mathcal{L}$ satisfies the *independence axiom* if for all three lotteries $L, L', L'' \in \mathcal{L}$ and for all $\alpha \in [0,1]$, we have

     $$L \succsim L' \text{ if and only if } \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

     That is, if we mix each of two lotteries with a third one, then the preference ordering of the two resulting lotteries does not depend on (is independent of) the particular third lottery we use.

   - *Nonparallel indifference curves.* In order to show that when indifference curves are not parallel straight lines (as we can see in figure 1) the independence axiom is violated, we need to show that

     $$L \succsim L' \text{ does } \textit{not} \text{ imply } \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

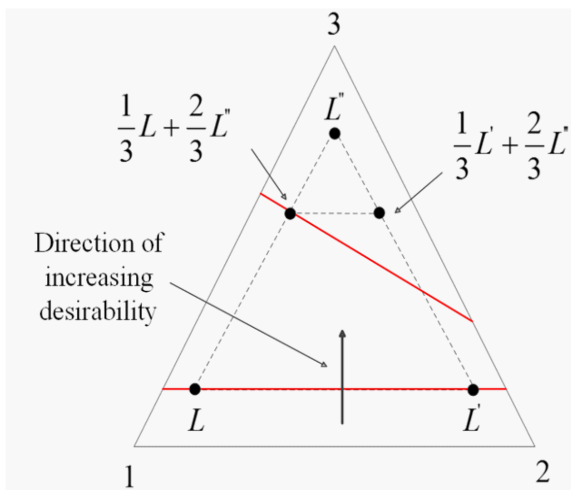

Figure 1. Nonparallel indifference curves.

This is easy to prove for the case in which the individual is indifferent between lotteries L and L', $L \sim L'$. That is, we must show

$$L \sim L' \text{ does } \textit{not} \text{ imply } \alpha L + (1-\alpha) L'' \sim \alpha L' + (1-\alpha) L''$$

Take the case in which $L \sim L'$, as lotteries L and L' in Figure 1, which lie on the same indifference curve. As we can see, the mix of each of these two lotteries with a third lottery L'' leads to compound lotteries $\frac{1}{3}L + \frac{2}{3}L''$ and $\frac{1}{3}L' + \frac{2}{3}L''$, respectively. The possibility of indifference curves which are *not* parallel lines leads to situations like that in the figure, where

$$\frac{1}{3}L + \frac{2}{3}L'' \prec \frac{1}{3}L' + \frac{2}{3}L''$$

since the decision maker prefers the second compound lottery to the first. Hence, we cannot guarantee that $L \sim L^{'}$ implies

$$\alpha L + (1-\alpha)\, L'' \sim \alpha L' + (1-\alpha)\, L''$$

As a consequence, the independence axiom does not hold for indifference curves which are not parallel lines.

2. Consider an individual with preferences over lotteries that satisfy the independence axiom. Answer the following questions.

   (a) Show that the independence axiom implies convexity, i.e., for three different lotteries L, L' and L'', if $L \succ L'$ and $L \succ L''$, then $L \succ \alpha L' + (1-\alpha)\, L''$.

   - From $L \succ L'$ we can apply the independence axiom, and obtain

     $$\alpha L + (1-\alpha)L \succ \alpha L' + (1-\alpha)L$$

     where note that we added $(1-\alpha)L$ on both sides of $L \succ L'$. Similarly, from $L \succ L''$ we can apply the independence axiom to obtain

     $$(1-\alpha)L + \alpha L' \succ (1-\alpha)L'' + \alpha L'$$

     where we added $\alpha L'$ on both sides of the strict preference relationship $L \succ L''$. By transitivity (from the two previous expressions), we have

     $$\alpha L + (1-\alpha)L \succ (1-\alpha)L'' + \alpha L'$$

     and rearranging

     $$L \succ \alpha L' + (1-\alpha)L''$$

     Intuitively, convex preference over lotteries means that if a decision maker prefers a lottery L over either two lotteries, $L^{'}$ or $L^{''}$, then he must also prefer lottery L over a convex combination of these two lotteries, $\alpha L' + (1-\alpha)L''$, i.e., the compound lottery of L' and L''.

(b) Discuss why a decision maker whose preferences violate convexity can be offered a sequence of choices that lead him to a sure loss of money

- If a decision maker's preferences over lotteries violate convexity, then we must have that for three different lotteries L, L' and L'', where $L \succsim L'$ and $L \succsim L''$, we obtain the opposite result than above; that is

$$\alpha L' + (1-\alpha)\, L'' \succ L$$

Note that, if the decision maker initially owns the right to participate in lottery L, he will be willing to pay an amount $\$X$ in order to switch to the compound lottery $\alpha L' + (1-\alpha)\, L''$ given that $\alpha L' + (1-\alpha)\, L'' \succ L$. Now he owns the compound lottery $\alpha L' + (1-\alpha)\, L''$, and either lottery L' or lottery L'' are realized. But we know that the decision maker prefers lottery L to either of these lotteries since

$$L \succsim L' \text{ and } L \succsim L''$$

was an initial assumption of this decision maker's preferences over lotteries. Therefore, he would be willing to pay again $\$Y$ in order to obtain lottery L. Hence, the decision maker is exactly as at the starting point of this sequence of deals (lottery L) and has lost $\$X + \$Y$. We can then repeat the process again and again, and make this individual pay $\$X + \$Y$ dollars, keeping him exactly where he started! Essentially, this type of decision maker could be subject to a systematic explanation (the so-called Dutch books), being wiped out of the market place.

3. Consider an old professor who is planning to retire next year. He is evaluating risky investments relative to the amount he invests (which we refer to as his reference point, r). Although he never studied prospect theory, his preferences might reveal that he compares investments according to this theory. In particular, for a return x on his investment r, his utility is $100(x-r) - \frac{1}{2}(x-r)^2$ if his return exceeds his investment, $x \geq r$, but becomes $400(x-r) + 2(x-r)^2$ if his return is smaller than his investment, $x < r$. Assume that his expected utility is linear in the probabilities. Intuitively, note that this individual's preferences exhibit loss aversion since the term $(x-r)$ is more heavily weighted when $x < r$ than otherwise.

(a) Assume that this individual plans to invest \$1,000 and faces two investment options: (1) bonds, that yield \$1,022.54 next year; and (2) stocks, that yield \$900 with probability 0.12 and \$1,100 with probability 0.88. Show that this individual is indifferent between both investment options, and he could thus opt for the riskless bonds.

- Investing \$1,000 in bonds yields a utility of

$$100(1022.54 - 1000) - \frac{1}{2}(1022.54 - 1000)^2 = 2000$$

since the constant return of the bond, $x = 1022.54$, exceeds the decision maker's initial investment (his reference point, $r = 1000$).

- Investing \$1,000 in stocks yields a utility of

$$\begin{aligned} & 0.12\left[400(900 - 1000) + 2(900 - 1000)^2\right] \\ & +0.88\left[100(1100 - 1000) - \frac{1}{2}(1100 - 1000)^2\right] \\ = \; & 0.12(-20000) + 0.88(5000) = 2000 \end{aligned}$$

since the stock yields a low return, $x = 900$, below the initial investment, $r = 1000$, with probability 0.12, but a high return $x = 1100$ above the initial investment $r = 1000$ with a probability 0.88. As a consequence, both investment options yield the same expected utility for this individual (2000).

(b) Generalize your previous result. That is, assume that the professor has an initial investment $I > 0$, and faces two investment options: (1) bonds yielding outcome I_b with certainty; and (2) stocks, yielding I_{SL} with probability p and I_{SH} with probability $1 - p$, where $I_{SH} > I > I_{LH} > 0$. Find under which values of p the equity premium puzzle emerges.

- In order to generalize our above results, the investment of I dollars in bonds yields a utility

$$100(I_b - I) - \frac{1}{2}(I_b - I)^2$$

since $I_b > I$ by definition. The investment in stocks yields an expected utility

$$p\left[400(I_{SL} - I) + 2(I_{SL} - I)^2\right] + (1 - p)\left[100(I_{SH} - I) - \frac{1}{2}(I_{SH} - I)^2\right]$$

simplifying,

$$100(I_{SH}-I)-\frac{1}{2}(I_{SH}-I)^2+p\left[400I_{SL}-100I_{SH}-300I+2(I_{SL}-I)^2+\frac{1}{2}(I_{SH}-I)^2\right]$$

For the professor to be indifferent, we must have his return from the bond be equal to the expected return from the stocks, i.e.,

$$\begin{aligned}
& 100(I_b-I)-\frac{1}{2}(I_b-I)^2 \\
= \; & 100(I_{SH}-I)-\frac{1}{2}(I_{SH}-I)^2 \\
& +p\left[400I_{SL}-100I_{SH}-300I+2(I_{SL}-I)^2+\frac{1}{2}(I_{SH}-I)^2\right]
\end{aligned}$$

Solving for p, we have

$$p=\frac{200(I_b-I_{SH})+(I_{SH}-I)^2-(I_b-I)^2}{800I_{SL}-200I_{SH}-600I+4(I_{SL}-I)^2+(I_{SH}-I)^2}$$

(c) Consider now an individual who does not use his initial investment as a reference point to evaluate future returns, i.e., $r=0$. Using the numerical values in part (a), does the equity premium puzzle still arise?

- For part (c), since we take $r=0$, the utility should always be evaluated in terms of $x\geq r$, for which $u(x)=100(x-r)-\frac{1}{2}(x-r)^2$, instead of $u(x)=400(x-r)+2(x-r)^2$. Even though the worse case scenario of \$900 is less than the initial investment of \$1,000, it is still greater than the point of evaluation at $r=0$. Therefore, the expected utility should be

$$\begin{aligned}
u(x) &= 0.12\left[100(900)-\frac{1}{2}(900)^2\right]+0.88\left[100(1100)-\frac{1}{2}(1100)^2\right] \\
&= 0.12*900(100-450)+0.88*1100(100-550) \\
&= -37800-435600 \\
&= -473400
\end{aligned}$$

which is more negative than that from investing in the bonds, with an expected utility $u_B=-420540.03$.

Therefore, the investor would still prefer to invest in bonds than stocks, even though he takes zero wealth, that is, $r=0$, as the reference point.