# Homework #5 (Due on 10/23/2025)

1. Assume that your utility function over income, $x$, is given by $u(x) = \sqrt{x}$, i.e., a Cobb-Douglas type of function. You have been offered two wage options.

   - In the first one you will receive a fixed salary of $54,000.
   - In the second one, you will only receive $4,000 as a fixed payment, plus a bonus of $100,000 if the firm is profitable. The probability that the firm goes profitable (and thus you get a total salary of $104,000) is 0.5, while the probability that the firm does not make enough profits is 0.5.

   (a) Find the expected value of the lottery induced by accepting the second wage offer.

   - The expected value of accepting the second wage offer is:

$$EV_{Second} = 0.5(\$4,000) + 0.5(\$104,000) = 2,000 + 52,000 = 54,000$$

   (b) Find the expected utility associated with the second offer.

   - The expected utility is

$$EU_{Second} = 0.5\sqrt{4,000} + 0.5\sqrt{104,000} = 192.87$$

   (c) Draw an approximate figure where the following elements are illustrated:

   1. Utility function (either concave, linear or convex);
   2. Utility level from the first wage offer;
   3. Utility level from each of the two possible outcomes of the second wage offer.
   4. Expected utility level from the second wage offer.

      - Figure 1 depicts the decision maker's concave utility function, the utility of the first (certain) wage offer of $\$54,000$ (232.38), the utility of the second (risky) wage offer by separately identifying the utility when the salary is only $\$4,000$ (63.25) and that when the worker receives the bonus, $\$104,000$ (322.5). The figure also depicts the expected utility from accepting the second wage offer, which is graphically illustrated by the mid point of the line connecting the utility in the case in which the decision maker only receives $\$4,000$ and when he receives $\$104,000$.

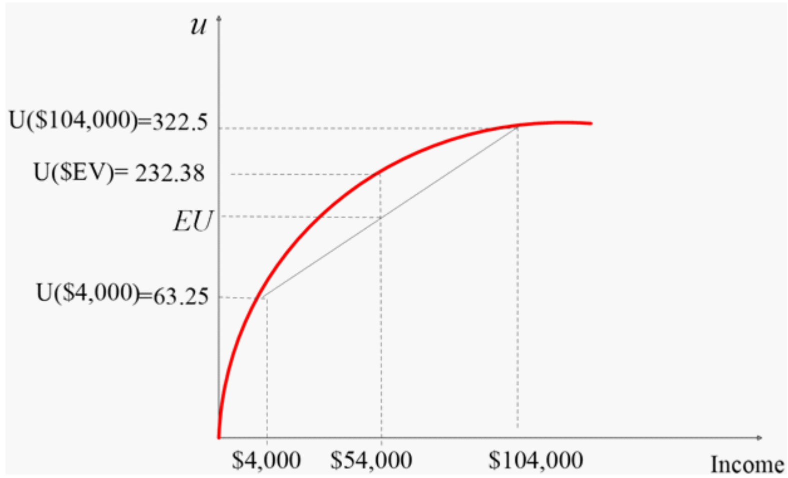


Figure 1. Utility function $u(x) = \sqrt{x}$

(d) Using your answers from parts (a) and (b), find the risk premium associated with the second offer.

- We know that the general expression of the risk premium ($RP$) of a lottery is

$$pu(x_1) + (1-p)u(x_2) = u(EV - RP)$$

Since the left hand side is just the expected utility from the lottery, $EU$, this expression can be move compactly written as

$$EU = u(EV - RP)$$

Given that we know $EU = 192.87$ from part (b) and that $EV = 54,000$ from part (a),

$$192.87 = \sqrt{54,000 - RP}$$

Squaring both sides of this equation and rounding to the nearest integer, yields

$$37,199 = 54,000 - RP \iff RP = \$16,801$$

In order to intuitively understand the risk premium of a lottery, consider a decision maker who is offered the expected value of the lottery ($\$54,000$), with an associated utility of 232.38 with certainty, or the possibility of playing the lottery (where he obtains an expected utility of 192.87). Needless to say, this risk averse individual would prefer the expected value of the lottery instead which, for convenience, coincides with the first wage offer. The risk premium hence measures by how much we need to reduce the certain wage offer of $\$54,000$ in order to make this individual become indifferent between a riskless offer (of $\$54,000 - \$16,801 = \$37,199$), or the expected utility of playing the lottery. In other words, a salary below $\$37,199$, despite being certain, would induce the individual decision maker to prefer the risky second wage offer.

(e) What amount of money should the first wage offer propose in order to make you indifferent between accepting the first and the second wage offers?

- The certain amount of money that would make you exactly indifferent between the utility from this certain payment and the utility from accepting the second (uncertain) wage offer is the so-called *Certainty Equivalent*. As described in our above discussion, the certainty equivalent is obtained from subtracting the risk premium to the certain amount (first wage offer),

$$Certainty\ Equivalent = \$54,000 - \$16,801 = \$37,199$$

(f) In your figure from part (c) include the risk premium and the certainty equivalent of the second wage offer.

- Graphically, the risk premium measures how much we need to move the first wage offer, $\$54,000$, leftward in figure 2 to make it map into the utility function at a height that exactly coincides with the expected utility of the lottery, denoted by $EU$ in figure 5.5. The wage level at which such coincidence occurs is the certainty equivalent of the lottery, $\$37,199$.

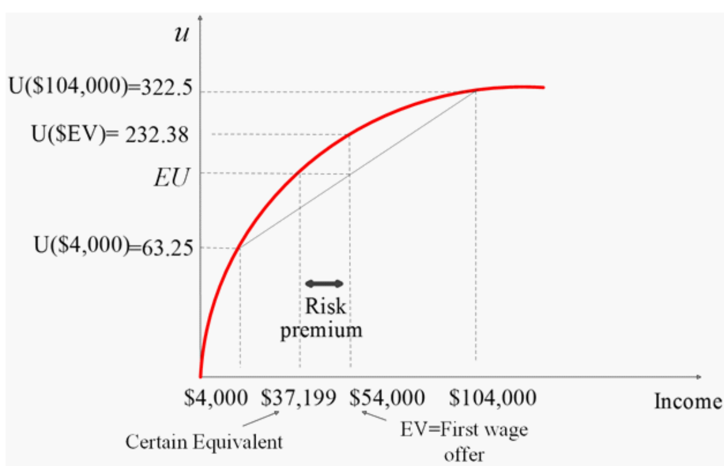


Figure 2. Risk premium of the lottery.

2. Consider the family of utility functions with Hyperbolic Absolute Risk Aversion (HARA)

as follows
$$u(x) = \frac{1}{\beta - 1}(\alpha + \beta x)^{\frac{\beta-1}{\beta}},$$
where $\beta \neq 0$ and $\beta \neq 1$. Find the Arrow-Pratt coefficient of absolute risk-aversion, $r_A(x, u)$.

- First, note that the first derivative of this utility function is $u'(x) = (\alpha + \beta x)^{-\frac{1}{\beta}}$, while the second derivative is $u''(x) = -(\alpha + \beta x)^{-\frac{1+\beta}{\beta}}$. Figure 3 depicts this function for different values of $\beta$.

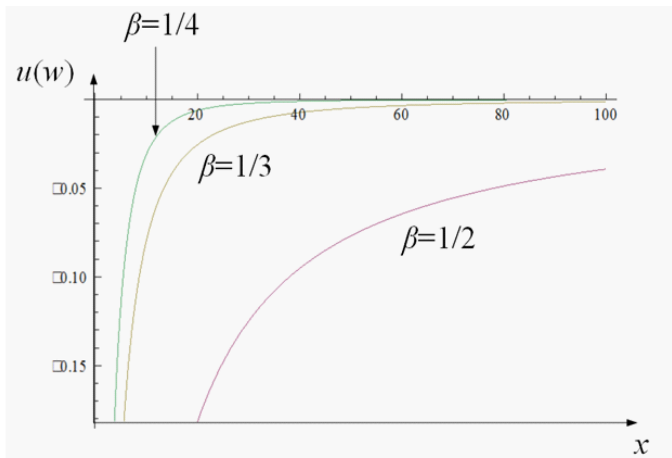


Figure 3. HARA utility function.

- The Arrow-Pratt coefficient of absolute risk-aversion is
$$r_A(x, u) = -\frac{u''(x)}{u'(x)} = -\frac{-(\alpha + \beta x)^{-\frac{1+\beta}{\beta}}}{(\alpha + \beta x)^{-\frac{1}{\beta}}} = \frac{1}{\alpha + \beta x},$$
which is decreasing in wealth, $x$, as long as $\beta > 0$, but it is increasing if $\beta < 0$. Figure 4 depicts the Arrow-Pratt coefficient of absolute risk aversion for different

values of parameter $\beta$.[1]

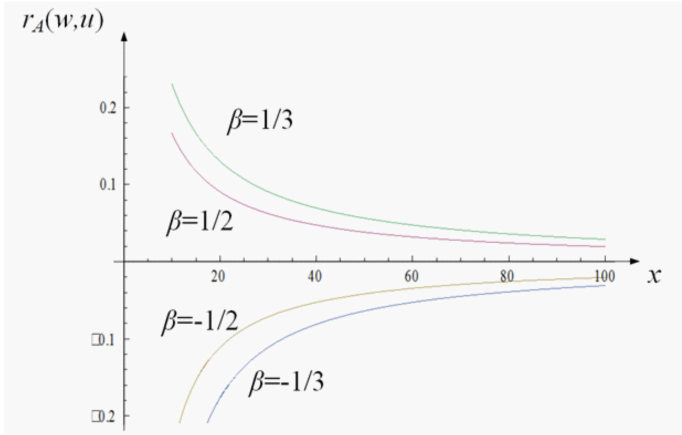


Figure 4. $r_A(x, u)$ for the HARA utility function.

3. An investor has the von Neumann Morgenstern utility function $u(c) = -e^{-\alpha c}$, where $c$ is consumption, and where $\alpha > 0$.[2] There are two states of the world, labelled 1 and 2, which are equally likely. There are two (rather extreme) assets, one of them attractive in state 1 and the other in state 2:

   - Asset 1 yields one unit of consumption in state 1, but nothing in state 2.
   - Asset 2 yields nothing in state 1, but one unit of consumption in state 2.
   - The price of the first asset is $\pi_1$, while the price of the second asset is $\pi_2$, where for simplicity $\pi_1 + \pi_2 = 1$. The investor starts with an endowment of $w$ units of both assets, but seeks to balance her portfolio so as to maximize her expected utility. Denote by $x_1$ the number of units that he acquires of the first asset, and by $x_2$ the number of units of the second asset.

   (a) Formulate the investor's expected utility maximization problem.

      – First, note that in state 1, the investor obtains one unit of consumption for every unit of asset $x_1$ he purchases, and no units of consumption for every unit of asset 2, i.e., $u(c) = -e^{-\alpha c} = -e^{-\alpha x_1}$. In state 2 the opposite happens, leading to a utility of $u(c) = -e^{-\alpha c} = -e^{-\alpha x_2}$. Since both states are equally likely, we can then express the investor's maximization

[1] For more information on the HARA utility function, including behavioral patterns in different investment settings, see its wikipedia entry at the following link: http://en.wikipedia.org/wiki/Hyperbolic_absolute_risk_aversion, and the references included in the link.

[2] Note that this utility function is increasing in consumption, i.e., $\frac{\partial u(c)}{\partial c} = \alpha e^{-\alpha c}$, which is positive for all $c > 0$; and concave since $\frac{\partial^2 u(c)}{\partial c^2} = -\alpha^2 e^{-\alpha c}$ is negative for all $c > 0$.

problem as choosing the levels of $x_1$ and $x_2$ that solve

$$\max_{x_1, x_2 \geq 0} \frac{1}{2} \cdot \left(-e^{-\alpha x_1}\right) + \frac{1}{2} \cdot \left(-e^{-\alpha x_2}\right)$$

$$\text{subject to } \pi_1 x_1 + \pi_2 x_2 \leq w$$

(b) Find the utility-maximizing purchases of assets 1 and 2, $x_1$ and $x_2$, for this investor.

– Setting up the Lagrangian associated to the above maximization problem,

$$\mathcal{L}(x_1, x_2; \lambda) = \frac{1}{2} \cdot \left(-e^{-\alpha x_1}\right) + \frac{1}{2} \cdot \left(-e^{-\alpha x_2}\right) + \lambda \left[w - \pi_1 x_1 - \pi_2 x_2\right]$$

and solving for the first order conditions,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\alpha}{2} e^{-\alpha x_1} - \lambda \pi_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\alpha}{2} e^{-\alpha x_2} - \lambda \pi_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= w - \pi_1 x_1 - \pi_2 x_2 = 0 \end{aligned}$$

Solving for $\lambda$ in the first two expressions, and setting them equal to each other, we obtain

$$\frac{e^{-\alpha x_1}}{\pi_1} = \frac{e^{-\alpha x_2}}{\pi_2} \Longleftrightarrow e^{-\alpha x_1 + \alpha x_2} = \frac{\pi_1}{\pi_2}$$

And applying logs,

$$-\alpha x_1 + \alpha x_2 = \ln \pi_1 - \ln \pi_2$$

Solving for $x_1$ on the binding budget constraint, we find $x_1 = \frac{w}{\pi_1} - \frac{\pi_2}{\pi_1} x_2$. Plugging this result into the above expression, yields

$$-\alpha \left(\frac{w}{\pi_1} - \frac{\pi_2}{\pi_1} x_2\right) + \alpha x_2 = \ln \pi_1 - \ln \pi_2$$

Multiplying both sides by $\frac{\pi_1}{\alpha}$,

$$x_2 \left[\pi_1 + \pi_2\right] = w + \frac{\pi_1}{\alpha} \left[\ln \pi_1 - \ln \pi_2\right]$$

and using the property that $\pi_1 + \pi_2 = 1$, then we obtain the optimal amount of asset $x_2$ that the investor demands,

$$x_2 = w + \frac{\pi_1}{\alpha} [\ln \pi_1 - \ln \pi_2]$$

Operating similarly, we can find the optimal amount of asset $x_1$,

$$x_1 = w + \frac{\pi_2}{\alpha} [\ln \pi_2 - \ln \pi_1]$$

(c) How does the holding of assets change with parameter $\alpha$? Interpret.

- The demand for assets 1 and 2 found above, $x_1$ and $x_2$, is *decreasing* in $\alpha$. It is easy to show that $\alpha$ is precisely the Arrow-Pratt coefficient of absolute risk-aversion of this investor,

$$r_A(x, u) = -\frac{u''(x)}{u'(x)} = \alpha,$$

which is constant for all wealth levels. Hence, an increase in his risk aversion (measured by $\alpha$) reduces his demand for any of these two risky assets.

(d) How does the investor's risk aversion and wealth level interact? How sensitive is this result to the specification of the utility function?

- As the previous section pointed out, the investor's demand for risky assets is independent on his wealth level. This result, however, depends on the particular specification of the investor's utility function. In this case, his utility function has an Arrow-Pratt coefficient of absolute risk-aversion, $r_A(x, u) = \alpha$, which is constant for all wealth levels. Many other utility function can be used which do not satisfy this property, and where the investor would vary his holdings of risky assets when he becomes more risk-averse.

4. Max Pullman lives for exactly two periods, $t = 0, 1$. Let $c_t \in \mathbb{R}$ denote his consumption in period $t$. Max's preferences (evaluated at $t = 0$) over two-period consumption streams are represented by function

$$U(c_0, c_1) = u(c_0) + \delta E u(c_1)$$

where $\delta$ is a discount factor, $u(\cdot)$ is an increasing and strictly concave utility function, and the $E$ operator denotes his expectation (at $t = 0$) concerning events in period $t = 1$. For simplicity, you can also assume that the marginal utility of consumption is convex, that is, $u''' > 0$.

Suppose that there is initially no uncertainty. Let $w_0 \geq 0$ be Max's income in period 0 and let $w_1 \geq 0$ denote his income in period 1. Max can save or borrow. Let $s \in \mathbb{R}$ denote his saving (notice that $s$ could be negative if he borrows) and let $\rho$ denote the gross return on saving (i.e., $\rho = 1 + r$ where $r$ is the interest rate). Thus, his consumption in period 0 is $w_0 - s$ and his consumption in period 1 is $w_1 + \rho s$. Assume interior solutions throughout the exercise.

(a) Write down necessary and sufficient conditions for Max's choice of saving, $s^*$, to be positive.

- Max's utility maximization problem is to choose $c_0$ and $c_1$ to solve

$$\begin{aligned} &\max_{c_0, c_1} u(c_0) + \delta u(c_1) \\ \text{subject to } c_0 &= w_0 - s \text{ and } c_1 = w_1 + \rho s \end{aligned}$$

  We can alternatively use the binding constraints to simplify the maximization problem into an unconstrained problem with a single choice variable, as follows,

$$\max_s u(w_0 - s) + \delta u(w_1 + \rho s)$$

  Taking first-order conditions with respect to $s$, we obtain

$$-u'(w_0 - s^*) + \delta \rho u'(w_1 + \rho s^*) = 0$$

  Hence, for $s^* > 0$, we require that $-u'(w_0) + \delta \rho u'(w_1) > 0$, as depicted in the vertical intercept of figure 5. The figure also shows that expression $-u'(w_0 - s^*) + \delta \rho u'(w_1 + \rho s^*)$ decreases in $s$, which is guaranteed by the strict concavity of $u(\cdot)$. (Finally, note that expression $-u'(w_0 + s^*) + \delta \rho u'(w_1 + \rho s^*)$ decreases at a decreasing rate, becoming flatter in $s$ as depicted in the figure, if $u''' > 0$.)

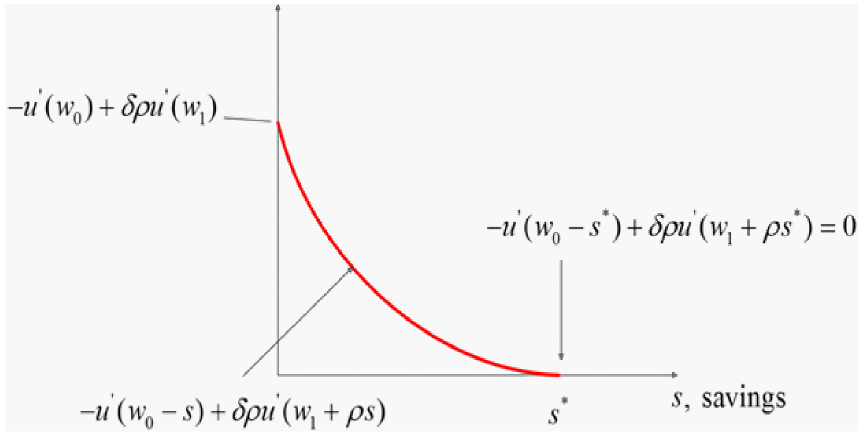


Figure 5. Positive savings $s^* > 0$.

- Therefore, since the vertical intercept satisfies $-u'(w_0) + \delta\rho u'(w_1) > 0$ then, solving for $\rho$ we obtain

$$\rho > \frac{u'(w_0)}{\delta u'(w_1)}.$$

- **Intuition:** The gross return on saving $\rho$ (the rate at which Max can transfer consumption from today to tomorrow) must be greater than his marginal rate of substitution evaluated at the endowment $(w_0, w_1)$ (that is, the rate at which he is willing to transfer consumption from today to consumption tomorrow). Otherwise, it would not be profitable for him to save (he would prefer to borrow).

(b) Suppose that $w_1 = 0$ and that the conditions you found in part $(a)$ hold. Find a condition on Max's coefficient of relative risk aversion that is necessary and sufficient for $s^*$ to be (locally) increasing in $\rho$.

- Evaluating the first order condition we found in part(a) at $w_1 = 0$, we have

$$\delta\rho u'(\rho s^*) = u'(w_0 - s^*)$$

Differentiating this first-order condition with respect to $\rho$ we obtain

$$\delta\left[u'(\rho s^*) + u''(\rho s^*)\left(\rho s^* + \rho^2\frac{\partial s^*}{\partial\rho}\right)\right] = -u''(w_0 - s^*)\frac{\partial s^*}{\partial\rho}$$

and rearranging,

$$\left[-u''(\rho s^*)\rho^2\delta - u''(w_0 - s^*)\right]\frac{\partial s^*}{\partial\rho} = \delta\left[u''(\rho s^*)(\rho s^*) + u'(\rho s^*)\right]$$

Dividing both sides of this equality by $u'(w_0 - s^*)$ yields

$$\left[-\frac{u''(\rho s^*)\rho^2\delta}{u'(w_0 - s^*)} - \frac{u''(w_0 - s^*)}{u'(w_0 - s^*)}\right]\frac{\partial s^*}{\partial \rho} = \delta\left[\frac{u''(\rho s^*)(\rho s^*)}{u'(w_0 - s^*)} + \frac{u'(\rho s^*)}{u'(w_0 - s^*)}\right]$$

Furthermore, from the first order condition we know that $u'(w_0 - s^*) = \delta\rho u'(\rho s^*) > 0$. Hence, the above expression can be rewritten as:

$$\begin{aligned}\left[-\frac{u''(\rho s^*)\rho^2\delta}{\delta\rho u'(\rho s^*)} - \frac{u''(w_0 - s^*)}{u'(w_0 - s^*)}\right]\frac{\partial s^*}{\partial \rho} &= \delta\left[\frac{u''(\rho s^*)(\rho s^*)}{\delta\rho u'(\rho s^*)} + \frac{u'(\rho s^*)}{\delta\rho u'(\rho s^*)}\right] \\ &= \frac{1}{\rho}\left[\frac{u''(\rho s^*)(\rho s^*)}{u'(\rho s^*)} + 1\right]\end{aligned}$$

In addition, since the term on the left-hand side is positive, i.e., $-\frac{u''(\rho s^*)\rho^2\delta}{\delta\rho u'(\rho s^*)} - \frac{u''(w_0 - s^*)}{u'(w_0 - s^*)} > 0$, given that $u' > 0$ and $u'' < 0$, a necessary and sufficient condition for $\frac{\partial s^*}{\partial \rho} > 0$ is that the right-hand side of the equality is positive as well, that is

$$\frac{u''(\rho s^*)(\rho s^*)}{u'(\rho s^*)} + 1 > 0 \Longleftrightarrow -\frac{u''(\rho s^*)(\rho s^*)}{u'(\rho s^*)} < 1$$

- **Interpretation:** Hence, the Arrow-Pratt's coefficient of relative risk aversion, $r_R(x) = \frac{-u''(x)x}{u'(x)}$, must be lower than one. This property holds for most of the concave utility functions, e.g., for $u(x) = \sqrt{x}$ coefficient $r_R(x)$ becomes $r$

$$_R(x) = -\frac{\frac{1}{4x^{3/2}}}{\frac{1}{2x^{1/2}}} = -\frac{1}{2} < 1.$$

Intuitively, this condition states that for Max to increase his savings as a response of a larger return $\rho$, his utility function must be sufficiently concave.

(c) Now suppose that Max faces uncertainty over his period 1 income. Specifically, suppose that his period 1 income is given by $w_1 + \widetilde{x}$ where $w_1 \geq 0$ and random variable $\widetilde{x}$ exhibits an expected value of $E(\widetilde{x}) = 0$. Let $s^{**}$ denote Max's new optimal saving in this context. Show that $s^{**} > s^*$. [*Hint*: Suppose that $s^{**} = s^*$ and compare the first order conditions using Jensen's inequality.]

- Under uncertainty, the first order condition becomes

$$-u'(w_0 - s^{**}) + \delta\rho E[u'(w_1 + \widetilde{x} + \rho s^{**})] = 0$$

By Jensen's inequality, the strict convexity of $u'$ (i.e., $u''' > 0$) implies

$$-u'(w_0-s^{**})+\delta\rho E[u'(w_1+\widetilde{x}+\rho s^{**})] > -u'(w_0-s^{**})+\delta\rho u'(w_1+E(\widetilde{x})+\rho s^{*})s^{**})]$$

and, since $E(\widetilde{x}) = 0$, the right-hand side of the inequality can be more compactly expressed as

$$-u'(w_0 - s^{**}) + \delta\rho u'(w_1 + \rho s^{**})] = 0$$

which exactly coincides with the first-order condition of exercise (a) and, hence, it is equal to zero. Hence, the above inequality becomes

$$-u'(w_0 - s^{**}) + \delta\rho E[u'(w_1 + \widetilde{x} + \rho s^{**})] > 0$$

indicating that, at a level of savings $s^*$, the curve $-u'(w_0 - s^*) + \delta\rho E[u'(w_1 + \widetilde{x} + \rho s^*)]$ is still positive, i.e., it has not crossed the horizontal axis yet.

- Figure 5 illustrates this comparison.

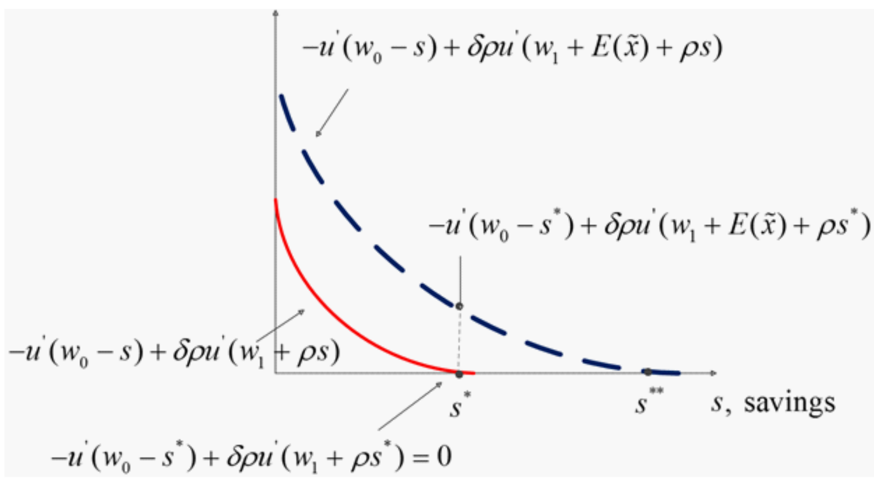


Figure 6. Comparing savings $s^*$ and $s^{**}$.

In particular, the figure depicts the curve $-u'(w_0-s^*)+\delta\rho u'(w_1+\rho s^*)]$, which we use in part (a) to identify the level of savings $s^*$, and curve $-u'(w_0 - s^*) + \delta\rho[u'(w_1 + E(\widetilde{x}) + \rho s^*)]$ that we use in part (c) to identify the level of savings $s^{**}$. Graphically, when evaluating both curves at the same level of savings $s^*$, we obtain that the latter is above the former, thus implying that $s^{**} > s^*$.