Homework #5 (Due on 10/23/2025)

1. Assume that your utility function over income, x, is given by $u(x) = \sqrt{x}$, i.e., a Cobb-Douglas type of function. You have been offered two wage options.

   - In the first one you will receive a fixed salary of \$54,000.
   - In the second one, you will only receive \$4,000 as a fixed payment, plus a bonus of \$100,000 if the firm is profitable. The probability that the firm goes profitable (and thus you get a total salary of \$104,000) is 0.5, while the probability that the firm does not make enough profits is 0.5.

   (a) Find the expected value of the lottery induced by accepting the second wage offer.

   (b) Find the expected utility associated with the second offer.

   (c) Draw an approximate figure where the following elements are illustrated:

      1. Utility function (either concave, linear or convex);
      2. Utility level from the first wage offer;
      3. Utility level from each of the two possible outcomes of the second wage offer.
      4. Expected utility level from the second wage offer.

   (d) Using your answers from parts (a) and (b), find the risk premium associated with the second offer.

   (e) What amount of money should the first wage offer propose in order to make you indifferent between accepting the first and the second wage offers?

   (f) In your figure from part (c) include the risk premium and the certainty equivalent of the second wage offer.

2. Consider the family of utility functions with Hyperbolic Absolute Risk Aversion (HARA) as follows

$$u(x) = \frac{1}{\beta - 1}(\alpha + \beta x)^{\frac{\beta-1}{\beta}},$$

   where $\beta \neq 0$ and $\beta \neq 1$. Find the Arrow-Pratt coefficient of absolute risk-aversion, $r_A(x, u)$.

3. An investor has the von Neumann Morgenstern utility function $u(c) = -e^{-\alpha c}$, where c is consumption, and where $\alpha > 0$.[1] There are two states of the world, labelled 1 and 2, which are equally likely. There are two (rather extreme) assets, one of them attractive in state 1 and the other in state 2:

[1] Note that this utility function is increasing in consumption, i.e., $\frac{\partial u(c)}{\partial c} = \alpha e^{-\alpha c}$, which is positive for all $c > 0$; and concave since $\frac{\partial^2 u(c)}{\partial c^2} = -\alpha^2 e^{-\alpha c}$ is negative for all $c > 0$.

- Asset 1 yields one unit of consumption in state 1, but nothing in state 2.
- Asset 2 yields nothing in state 1, but one unit of consumption in state 2.
- The price of the first asset is π_1, while the price of the second asset is π_2, where for simplicity $\pi_1 + \pi_2 = 1$. The investor starts with an endowment of w units of both assets, but seeks to balance her portfolio so as to maximize her expected utility. Denote by x_1 the number of units that he acquires of the first asset, and by x_2 the number of units of the second asset.

  (a) Formulate the investor's expected utility maximization problem.

  (b) Find the utility-maximizing purchases of assets 1 and 2, x_1 and x_2, for this investor.

  (c) How does the holding of assets change with parameter α? Interpret.

  (d) How does the investor's risk aversion and wealth level interact? How sensitive is this result to the specification of the utility function?

4. Max Pullman lives for exactly two periods, $t = 0, 1$. Let $c_t \in \mathbb{R}$ denote his consumption in period t. Max's preferences (evaluated at $t = 0$) over two-period consumption streams are represented by function

$$U(c_0, c_1) = u(c_0) + \delta E u(c_1)$$

where δ is a discount factor, $u(\cdot)$ is an increasing and strictly concave utility function, and the E operator denotes his expectation (at $t = 0$) concerning events in period $t = 1$. For simplicity, you can also assume that the marginal utility of consumption is convex, that is, $u''' > 0$.

Suppose that there is initially no uncertainty. Let $w_0 \geq 0$ be Max's income in period 0 and let $w_1 \geq 0$ denote his income in period 1. Max can save or borrow. Let $s \in \mathbb{R}$ denote his saving (notice that s could be negative if he borrows) and let ρ denote the gross return on saving (i.e., $\rho = 1 + r$ where r is the interest rate). Thus, his consumption in period 0 is $w_0 - s$ and his consumption in period 1 is $w_1 + \rho s$. Assume interior solutions throughout the exercise.

(a) Write down necessary and sufficient conditions for Max's choice of saving, s^*, to be positive.

(b) Suppose that $w_1 = 0$ and that the conditions you found in part (a) hold. Find a condition on Max's coefficient of relative risk aversion that is necessary and sufficient for s^* to be (locally) increasing in ρ.

(c) Now suppose that Max faces uncertainty over his period 1 income. Specifically, suppose that his period 1 income is given by $w_1 + \widetilde{x}$ where $w_1 \geq 0$ and random variable $\widetilde{x}$ exhibits an expected value of $E(\widetilde{x}) = 0$. Let s^{**} denote Max's new optimal saving in this context. Show that $s^{**} > s^*$. [*Hint*: Suppose that $s^{**} = s^*$ and compare the first order conditions using Jensen's inequality.]