Homework #4 (10/07/2025)

1. Suppose that a firm owns two plants, each producing the same good. Every plant j's average cost is given by

$$AC_j(q_j) = \alpha + \beta_j q_j \text{ for } q_j \geq 0, \text{ where } j = \{1, 2\}$$

where coefficient β_j may differ from plant to plant, i.e., if $\beta_1 > \beta_2$ plant 2 is more efficient than plant 1 since its average costs increase less rapidly in output. Assume that you are asked to determine the cost-minimizing distribution of aggregate output $q = q_1 + q_2$, among the two plants (i.e., for a given aggregate output q, how much q_1 to produce in plant 1 and how much q_2 to produce in plant 2.) For simplicity, consider that aggregate output q satisfies $q < \frac{\alpha}{\max_j |\beta_j|}$. (You will be using this condition in part b.)

(a) If $\beta_j > 0$ for every plant j, how should output be located among the two plants?

- The cost-minimization problem in which we find the optimal combination of output q_1 and q_2 that minimizes the total cost of production across plants is

$$\min_{q_1, q_2} \; TC_1(q_1) + TC_2(q_2)$$

$$\text{subject to } \; q_1 + q_2 = q$$

or equivalently, the profit maximization problem in which firms choose the optimal combination of output q_1 and q_2 that maximizes the total profits across all plants is

$$\max_{q_1, q_2} \; \underbrace{pq_1 - TC_1(q_1)}_{\pi_1} + \underbrace{pq_2 - TC_2(q_2)}_{\pi_2}$$

$$\text{subject to } \; q_1 + q_2 = q$$

- If the average cost is $AC_j(q_j) = \alpha + \beta_j q_j$ then the total cost is $TC_j(q_j) = (\alpha + \beta_j q_j)q_j$. Thus, we can rewrite the above PMP as:

$$\max_{q_1, q_2} \; pq_1 - (\alpha + \beta_1 q_1)q_1 + pq_2 - (\alpha + \beta_2 q_2)q_2$$

$$\text{subject to } \; q_1 + q_2 = q$$

Taking first order conditions with respect to q_1 and q_2 yields

$$\frac{\partial(\pi_1+\pi_2)}{\partial q_1} = p - \alpha - 2\beta_1 q_1 = \lambda$$

$$\frac{\partial(\pi_1+\pi_2)}{\partial q_2} = p - \alpha - 2\beta_2 q_2 = \lambda$$

$$\frac{\partial(\pi_1+\pi_2)}{\partial \lambda} = q_1 + q_2 = q$$

Using the first two order conditions, we obtain

$$p - \alpha - 2\beta_1 q_1 = p - \alpha - 2\beta_2 q_2$$

and rearranging, $q_2 = \frac{\beta_1}{\beta_2} q_1$. Replacing this expression into the constraint $q_1 + q_2 = q$ yields

$$q_1 + \underbrace{\frac{\beta_1}{\beta_2} q_1}_{q_2} = q$$

and solving for q_1 entails the cost-minimizing production in plant 1,

$$q_1\left(1 + \frac{\beta_1}{\beta_2}\right) = q, \quad \text{thus} \quad q_1 = \frac{\beta_2}{\beta_1+\beta_2} q,$$

and operating similarly for q_2, we find

$$q_2 = \frac{\beta_1}{\beta_1+\beta_2} q$$

- *Extension:* Note that, generally for J plants, the average cost of plant j is $AC_j(q_j) = \alpha + \beta_j q_j$ implying that the total cost must be $TC_j(q_j) = (\alpha + \beta_j q_j)q_j$. Therefore, plant j's marginal cost is $MC_j(q_j) = \alpha + 2\beta_j q_j$. Since $\beta_j > 0$ for every j, the first order necessary and sufficient conditions for cost minimization are: (1) that firms' marginal costs coincide (otherwise, we would still have incentives to distribute a larger production to those firms with the lowest marginal cost)

$$MC_j(q_j) = MC_{j'}(q_{j'}) \text{ for any two plants } j \text{ and } j'$$

and; (2) that the aggregate output constraint holds

$$q_1 + q_2 + ... q_J = q.$$

From these conditions we obtain

$$q_j = \frac{\frac{q}{\beta_j}}{\sum_h \frac{1}{\beta_h}}.$$

which coincides with our results for $N = 2$ plants,

$$q_1 = \frac{\frac{q}{\beta_1}}{\frac{1}{\beta_1} + \frac{1}{\beta_2}} = \frac{\beta_2}{\beta_1 + \beta_2} q.$$

Figure 1 depicts the average and marginal cost curves for two plants satisfying $\beta_2 > \beta_1$. In particular, the firm manager chooses, for a given aggregate output $q = q_1 + q_2$, the individual output levels q_1 and q_2 that equate the marginal costs across both plants (see vertical axis).

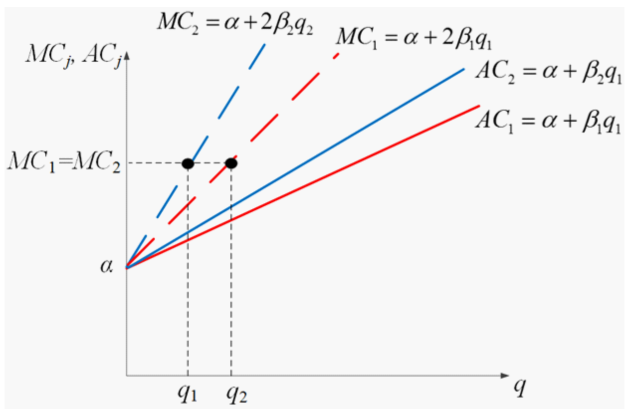

Figure 1. $\beta_j > 0$ for every firm.

(b) If $\beta_j < 0$ for every plant j, how should output be located among the two plants?

- First, note that $\beta_j < 0$ implies that the average cost $AC_j(q_j) = \alpha + \beta_j q_j$ is decreasing in output. Hence, it is cost-minimizing to concentrate all production on the plant with the smallest $\beta_j < 0$ (the most negative β_j) because average costs (and total costs) are minimized by doing so.
- Figure 2 depicts a firm in which both plants exhibit decreasing average costs, but $\beta_2 < \beta_1 < 0$, implying that it is beneficial for the firm to concentrate all output in plant 2. In addition, note that the average cost in plant 1 is positive for all q_1 as long as $\alpha - \beta_1 q_1 > 0$, or $q_1 < \frac{\alpha}{\beta_1}$, where $\frac{\alpha}{\beta_1}$ represents the horizontal intercept of AC_1 in the figure. Similarly for firm 2, where $AC_2 > 0$ for all q_2 as long as $q_2 < \frac{\alpha}{\beta_2}$, where $\frac{\alpha}{\beta_2}$ represents the horizontal intercept of AC_2. Hence, the original condition $q < \frac{\alpha}{\max_j |\beta_j|}$ is equivalent to $q < \min_j \frac{\alpha}{|\beta_j|}$,

graphically implying that the aggregate output q lies to the left-hand side to the smallest horizontal intercept.

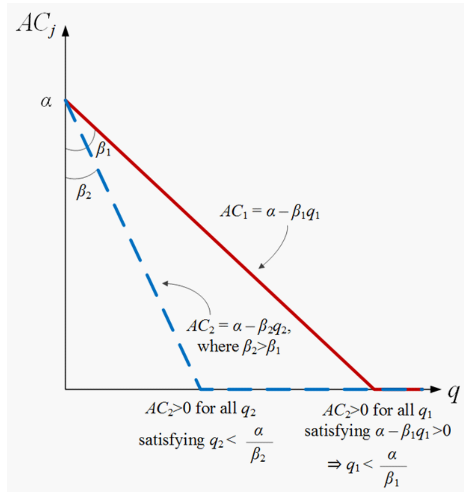

Figure 2. $\beta_j < 0$ for every firm.

(c) If $\beta_j > 0$ for some plants and $\beta_i < 0$ for others?

- Similarly as in part (b), the firm now faces some plants with increasing average costs (those with $\beta_j > 0$) and some plants with decreasing average costs (those with $\beta_j < 0$). Hence, it is cost-minimizing to concentrate all production on the plant/s with the smallest $\beta_j < 0$, since it benefits from the most rapidly decreasing average costs. Figure 3 depicts a firm with plant 1 (2) having increasing (decreasing, respectively) average costs.

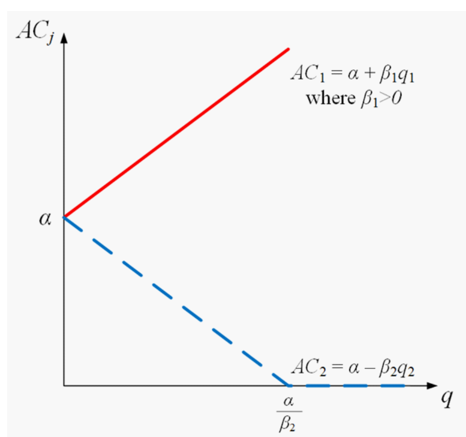

Figure 3. $\beta_1 > 0$ and $\beta_2 < 0$.

2. Consider a Cobb-Douglas production function $f : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$, given by $f(z) = 2^{3/4} z_1^{1/4} z_2^{1/4}$, where $z_1 \geqslant 0$ and $z_2 \geqslant 0$ denote inputs in the production process.

(a) Check if the production function has nonincreasing, nondecreasing, or constant returns to scale.

- *Nonincreasing returns to scale.* If the production function satisfies nonincreasing returns to scale, for all inputs $z \in \mathbb{R}^2_+$ and for all $\alpha > 1$, we must have $\alpha f(z) \geq f(\alpha z)$. Intuitively, increasing all inputs by a common factor α, yields a less-than-proportional increase in output, $f(\alpha z)$, i.e., $\alpha f(z) \geq f(\alpha z)$. In this exercise, this condition implies

$$\alpha \left(2^{3/4} z_1^{1/4} z_2^{1/4}\right) \geq 2^{3/4} \left(\alpha z_1\right)^{1/4} \left(\alpha z_2\right)^{1/4}$$

  Simplifying the right-hand side, we obtain

$$\alpha \left(2^{3/4} z_1^{1/4} z_2^{1/4}\right) \geq \alpha^{1/2} \left(2^{3/4} z_1^{1/4} z_2^{1/4}\right) \Leftrightarrow \alpha \geq \alpha^{\frac{1}{2}}$$

  which is satisfied for all $\alpha > 1$. Hence, this production function satisfies nonincreasing returns to scale.

- *Nondecreasing returns to scale.* (Since this production function exhibits nonincreasing returns to scale, and such property holds strictly, we can actually anticipate that it will not satisfy nondecreasing returns to scale. However, and as a practice, we go over these properties nevertheless.) If the production function satisfies nondecreasing returns to scale, for all inputs $z \in \mathbb{R}^2_+$ and for all $\alpha > 1$, we must have $\alpha f(z) \leq f(\alpha z)$. In this case, a common increase in all inputs by a common factor α, yields a more-than-proportional increase in output, $f(\alpha z)$, i.e., $\alpha f(z) \leq f(\alpha z)$. In this exercise, this condition implies

$$\alpha \left(2^{3/4} z_1^{1/4} z_2^{1/4}\right) \leq \alpha^{1/2} \left(2^{3/4} z_1^{1/4} z_2^{1/4}\right) \Leftrightarrow \alpha \leq \alpha^{\frac{1}{2}}$$

  and this inequality cannot hold for any $\alpha > 1$. Then, this production function cannot exhibit nondecreasing returns to scale.

- *Constant returns to scale.* If the production function satisfies constant returns to scale, it must satisfy nonincreasing and nondecreasing returns to scale. Since this production function does not satisfy both, it cannot exhibit constant returns to scale. In particular, when a production function exhibits constant returns to scale, a common increase of all inputs by a common factor $\alpha > 1$, yields a proportional increase in output, $f(\alpha z)$, i.e., $\alpha f(z) = f(\alpha z)$.

- *Remark*: This production function is a standard Cobb-Douglas production function $f(z) = A z_1^{\alpha} z_2^{\beta}$. It is good to remember that when $\alpha + \beta \leq 1$ the production function has nonincreasing returns to scale, when $\alpha + \beta \geq 1$ it has nondecreasing returns to scale, and when $\alpha + \beta = 1$ it exhibits constant returns to scale.

(b) Let $w \in \mathbb{R}^2_{++}$ denote the vector of input prices and $p > 0$ the output price. Determine for each output level $q \geqslant 0$ the cost function $c(w, q)$ and the conditional factor demand $z(w, q)$.

- We first need to find the conditional factor demand (solving the cost minimization problem, CMP, of the firm), and afterwards we can compute the firm's cost function, as the value function emerging from the CMP.

$$\text{CMP} : \min_{z \geq 0} \; w \cdot z$$

$$\begin{aligned} \text{subject to} \quad 2^{3/4} z_1^{1/4} z_2^{1/4} &\geq q, \\ z &\geq 0 \end{aligned}$$

First, note that $z = 0$ can be ruled out. Indeed, from the production function we know that output would be zero when either of the inputs are zero, i.e., $z_1 = 0$ or $z_2 = 0$. Secondly, the production function constraint $2^{3/4} z_1^{1/4} z_2^{1/4} \geq q$ must be binding since the production function is strictly increasing in both inputs and inputs are costly (they are not free since the vector of input prices $w \in \mathbb{R}^2_{++}$ is strictly positive in all components).[1] Thus, we can solve for z_1 in this constraint, finding

$$2^{3/4} z_1^{1/4} z_2^{1/4} = q \Longleftrightarrow z_1 = \frac{1}{8} \frac{q^4}{z_2}.$$

We can now substitute this result into the previous cost minimization problem, thus reducing the number of choice variables to only one, z_2, as follows

$$\min_{z_2} \; w_1 \cdot z_1 + w_2 \cdot z_2 = w_1 \cdot \left(\frac{1}{8} \frac{q^4}{z_2} \right) + w_2 \cdot z_2$$

The first order condition with respect to z_2 is

$$-w_1 \left(\frac{1}{8} \frac{q^4}{z_2^2} \right) + w_2 = 0$$

[1] Alternatively, one can set up the Lagrangian of the firm's profit maximization problem (PMP) using λ as the Lagrange multiplier of constraint $2^{3/4} z_1^{1/4} z_2^{1/4} \geq q$, then take first-order conditions with respect to inputs z_1 and z_2, and obtain that $\lambda > 0$, which implies that the above constraint holds with equality.

and solving for z_2, yields

$$z_2 = \frac{1}{2}q^2\sqrt{\frac{w_1}{2w_2}}$$

Substituting z_2 into the expression for z_1 we found above, we obtain

$$z_1 = \frac{1}{8}\frac{q^4}{z_2} \Longrightarrow z_1 = \frac{1}{8}\frac{q^4}{\left(\frac{1}{2}q^2\sqrt{\frac{w_1}{2w_2}}\right)} = \frac{1}{2}q^2\sqrt{\frac{w_2}{2w_1}}$$

Therefore, the conditional factor demand is

$$z(w,q) = \left(\frac{1}{2}q^2\sqrt{\frac{w_2}{2w_1}}, \frac{1}{2}q^2\sqrt{\frac{w_1}{2w_2}}\right)$$

As a consequence, the cost function (i.e., the minimal cost that the firm must incur in order to attain output level of q) is

$$\begin{aligned} c(w,q) &= w_1 \cdot z_1(w,q) + w_2 \cdot z_2(w,q) \\ &= w_1\frac{1}{2}q^2\sqrt{\frac{w_2}{2w_1}} + w_2\frac{1}{2}q^2\sqrt{\frac{w_1}{2w_2}} \\ &= \frac{1}{2}q^2\sqrt{2w_1w_2} \end{aligned}$$

which can be interpreted as the value function of the CMP, since we evaluated the objective function at the arguments that solved the CMP.

(c) Verify Shephard's lemma.

- Let us first recall Shephard's lemma: If the production set is *closed* and satisfies the *free-disposal* property, and the conditional factor demand $z(\overline{w}, q)$ consists of a single point $\overline{z}$, then the cost function $c(w,q)$ is differentiable with respect to w at $\overline{w}$, and this derivative is

$$\frac{\partial c(\overline{w}, q)}{\partial w_l} = \overline{z}_l$$

  Hence, in order to verify Shephard's lemma, we must first check that the production set Y is closed, and that it satisfies the free disposal property.

- *Closedness.* The production set associated with the production function is given by

$$Y = \left\{(-z,q) \in \mathbb{R}^3 : q \leq f(z) \text{ and } z \in \mathbb{R}^2_+\right\}$$

and for convenience, we can rewrite this set as

$$Y = \left\{y \in \mathbb{R}^3 : y_1 \le 0\right\} \cap \left\{y \in \mathbb{R}^3 : y_2 \le 0\right\} \cap \left\{y \in \mathbb{R}^3 : y_3 \le f(-y_1, -y_2)\right\}$$

which is the intersection of three closed sets (the first two representing inputs, and the third representing output), and as a consequence it is closed. [Recall that the intersection of finitely many closed sets is also closed.]

- *Free-disposal.* Consider two input-output pairs that belong to production set Y, $(-z, q) \in Y$ and $(-z', q')$, where $(-z', q') \le (-z, q)$, as depicted in the figure below. This means that the second pair either uses more inputs as the first pair (producing the same output) or uses the same amount of inputs (but produces a smaller output). That is, either: (1) $z_1' \ge z_1$ or $z_2' \ge z_2$ but producing the same output $q' = q$; or (2) $z_1' = z_1$ and $z_2' = z_2$ but producing less output $q' \le q$. In order to show that the free-disposal property is satisfied, we must show that $(-z', q')$ also belongs to the production set Y. Since the production function $f(\cdot)$ is weakly increasing in both inputs z_1 and z_2, we find that

$$q' \le q \le f(z) \le f(z').$$

  That is, $(-z', q')$ also belongs to the production set Y.

- Hence, the production set Y is closed and satisfies free-disposal, implying that all conditions for Shephard's lemma hold.[2] We can thus determine the conditional factor demand function for input 1, $z_1(w, q)$, by differentiating the cost function, $c(w, q)$, with respect to the price of input 1, as follows

$$\frac{\partial c(w,q)}{\partial w_1} = \frac{\partial \left(\frac{1}{2}q^2\sqrt{2w_1w_2}\right)}{\partial w_1} = \frac{1}{2}q^2\sqrt{\frac{w_2}{2w_1}}$$

  and similarly for the conditional factor demand of input 2, $z_2(w, q)$,

$$\frac{\partial c(w,q)}{\partial w_2} = \frac{\partial \left(\frac{1}{2}q^2\sqrt{2w_1w_2}\right)}{\partial w_2} = \frac{1}{2}q^2\sqrt{\frac{w_1}{2w_2}}$$

(d) Determine the profit function $\pi(p, w)$.

- To determine the profit function, we can solve the profit maximization prob-

[2] Note that there is an additional condition, which states that conditional factor demand correspondences consist of a single point (they are functions); and this was clearly satisfied in our exercise. For a given input price vector $w = (w_1, w_2)$ and output q, the function $z(w, q)$ yields a real number for the input usage of z_1 and another for z_2.

lem using the cost function,

$$\max_{q\geq 0} \; pq - \frac{1}{2}q^2\sqrt{2w_1w_2}$$

Taking first-order conditions with respect to q yields

$$p - q^*\sqrt{2w_1w_2} \leq 0$$

which holds with equality in interior solutions, $q^* > 0$. In the case of interior solutions, we can solve for q^* to obtain the following profit-maximizing output

$$q^* = \frac{p}{\sqrt{2w_1w_2}}.$$

And the profit arising from producing this output level is

$$\begin{aligned} \pi(p, w) &= pq^* - \frac{1}{2}(q^*)^2\sqrt{2w_1w_2} \\ &= \frac{p^2}{2\sqrt{2w_1w_2}} \end{aligned}$$

Again we can see that since $p > 0$ and $w > 0$, the profit from producing q^* is positive for all parameter values. It is therefore never optimal to remain inactive, i.e., set $q^* = 0$ (which gives zero profits).

- *Sufficiency*: Let us now check second order conditions. The above PMP is striclty concave, and thus the output level q^* that we found is profit maximizing, if the cost function is convex in q, which holds in this case since

$$\frac{\partial c(w,q)}{\partial q} = q\sqrt{2w_1w_2} \text{ and } \frac{\partial^2 c(w,q)}{\partial q^2} = \sqrt{2w_1w_2} > 0$$

for all $w_1, w_2 > 0$.

3. Consider a firm whose production function $f(z)$ exhibits constant returns to scale. Show that its cost function can be expressed as $c(w,q) = q \cdot c(w,1)$, i.e., the cost per unit times the number of units produced.

   - By the definition of constant returns to scale, a change in input usage of αz where $\alpha > 0$, entails a change in output of αq, and viceversa (if we plan to vary output by αq then we need to change the amount of inputs being used by αz). As a consequence, changing output by αq entails that the cost function satisfies $c(w, \alpha q) = \alpha \cdot c(w,q)$, i.e., initial costs change by exactly α. If $\alpha = \frac{1}{q}$, we obtain that

$$c(w, \frac{1}{q}q) = \frac{1}{q} \cdot c(w, q)$$

which simplifies to

$$c(w, 1) = \frac{1}{q} \cdot c(w, q)$$

Multiplying both sides by q, we obtain

$$q \cdot c(w, 1) = c(w, q)$$

where $c(w, 1)$ denotes the cost of producing one unit (average cost).

4. Show that, if a production function $f : \mathbb{R}^{L-1} \to \mathbb{R}$ satisfies increasing returns to scale, that is,

$$\text{for every } z \in \mathbb{R}^{L-1} \text{ and for every } t \geqslant 1,\ f(tz) \geqslant t\ f(z)$$

then $f(z)$ also satisfies *increasing average product* property.

- Let the common increase in all inputs, t, be $t = \frac{z'}{z}$, where $z' \geq z$. We therefore have $t \geq 1$ and, plugging $t = \frac{z'}{z}$ into the definition of increasing returns to scale, we obtain

$$f\left(\frac{z'}{z}z\right) \geqslant \frac{z'}{z}\ f(z)$$

multiplying both sides now by $\frac{1}{z'}$, yields

$$\frac{1}{z'}f\left(\frac{z'}{z}z\right) \geqslant \frac{1}{z'}\frac{z'}{z}\ f(z)$$

which simplifies to

$$\frac{f(z')}{z'} \geqslant \frac{f(z)}{z}$$

which exactly represents increasing *average* product. Hence, increasing returns to scale imply increasing average product.