Recitation (09/19/2025)

1. Show that the compensating and the equivalent variation coincide when the utility function is quasilinear with respect to the first good (and we fix $p_1 = 1$). [*Hint*: Use the definitions of the compensating and equivalent variations in terms of the expenditure function (not the hicksian demand). In addition, recall that if $u(x)$ is quasilinear with respect to good 1, then we can express it as

$$u(x) = x_1 + \phi\left(x_{-1}\right),$$

where x_{-1} represents all the remaining goods, $l = 2, 3, ..., L$.]

From the definition of the compensating and the equivalent variation, we know that

$$\begin{aligned} CV\left(p^0, p^1, w\right) &= e\left(p^1, u^1\right) - e\left(p^1, u^0\right) \\ EV\left(p^0, p^1, w\right) &= e\left(p^0, u^1\right) - e\left(p^0, u^0\right) \end{aligned}$$

Moreover, we know that if $u(x)$ is quasilinear with respect to good 1, then we can express it as

$$u(x) = x_1 + \phi\left(x_{-1}\right) \Longleftrightarrow x_1 = u(x) - \phi\left(x_{-1}\right)$$

where x_{-1} represents all the remaining goods, $l = 2, 3, ..., L$. Therefore, the expenditure function becomes

$$e\left(p, u\right) = \sum_{i=1}^{L} p_i x_i = \underbrace{p_1}_{\$1} x_1 + \underbrace{\sum_{k=2}^{L} p_k x_k}_{p_{-1} \cdot x_{-1}} = x_1 + p_{-1} \cdot x_{-1}$$

where we use the fact that $p_1 = 1$. Substituting x_1 from the above expression of the quasi-linear utility function, we have

$$e\left(p, u\right) = \underbrace{u(x) - \phi\left(x_{-1}(p_{-1})\right)}_{x_1} + p_{-1} \cdot x_{-1}$$

Using this expression for the expenditure function into the above definition of the compensating variation, we obtain

$$\begin{aligned} CV\left(p^0, p^1, w\right) &= e\left(p^1, u^1\right) - e\left(p^1, u^0\right) \\ &= \left[u^1(x) - \phi\left(x_{-1}(p_{-1})\right) + p_{-1}\cdot x_{-1}\right] \\ &\quad - \left[u^0(x) - \phi\left(x_{-1}(p_{-1})\right) + p_{-1}\cdot x_{-1}\right] \\ &= u^1(x) - u^0(x) \end{aligned}$$

And similarly for the definition of the equivalent variation,

$$\begin{aligned} EV\left(p^0, p^1, w\right) &= e\left(p^0, u^1\right) - e\left(p^0, u^0\right) \\ &= \left[u^1(x) - \phi\left(x_{-1}(p_{-1})\right) + p_{-1}\cdot x_{-1}\right] \\ &\quad - \left[u^0(x) - \phi\left(x_{-1}(p_{-1})\right) + p_{-1}\cdot x_{-1}\right] \\ &= u^1(x) - u^0(x) \end{aligned}$$

Therefore, for quasilinear utility functions, the compensating and the equivalent variation give us the *same* measures of the monetary value, $u^1(x) - u^0(x)$, that a consumer would assign to a reduction (or increase) in the price of a good. Recall that this is because quasilinear utility functions do not generate income effects, and when income effects are absent, the equivalent variation coincides with the compensating variation, and they both coincide with the change in consumer surplus.

1. Consider the following profit function that has been obtained from a technology that uses a single input, z:

$$\pi(p, w) = p^2 w^\alpha$$

   where p is the output price, w is the input price and α is a parameter value.

   (a) Check if the profit function satisfies homogeneity of degree one jointly in both p and w. In particular, determine for which values of α this property is satisfied.

   - The profit function is homogeneous of degree one if

$$\pi(\theta p, \theta w) = \theta\pi(p, w)$$

     In this case we have that the left-hand term becomes

$$\pi(\theta p, \theta w) = (\theta p)^2(\theta w)^\alpha = \theta^{2+\alpha}p^2 w^\alpha \tag{3}$$

and, on the other hand, the right-hand term is

$$\theta\pi(p,w) = \theta p^2 w^\alpha \tag{4}$$

since, by homogeneity of degree one, expressions (3) and (4) must coincide. Then,

$$\theta^{2+\alpha} p^2 w^\alpha = \theta p^2 w^\alpha$$

which implies that $2+\alpha = 1$. That is, we need $\alpha = -1$. As a consequence, the profit function that we obtain is

$$\pi(p,w) = \frac{p^2}{w}$$

(b) Assuming the value of α for which the profit function satisfies homogeneity of degree one, check if the profit function $\pi(p,w)$ satisfies the following properties: (1) non-decreasing in output price p, (2) non-increasing in input prices w, and (3) convex in prices p and w.

- *Non-decreasing in the output price, p:* Increasing output prices yields a weakly higher profit level since
$$\frac{\partial \pi(p,w)}{\partial p} = \frac{2p}{w} \geq 0$$
- *Non-increasing in the factor prices, w:* Increasing all input prices weakly reduces profits since
$$\frac{\partial \pi(p,w)}{\partial w} = -\frac{p^2}{w^2} \leq 0$$
- *Convex in prices* (factor prices and output prices):

$$\begin{vmatrix} \frac{\partial^2 \pi(p,w)}{\partial p^2} & \frac{\partial^2 \pi(p,w)}{\partial p \partial w} \\ \frac{\partial^2 \pi(p,w)}{\partial w \partial p} & \frac{\partial^2 \pi(p,w)}{\partial w^2} \end{vmatrix} = \begin{vmatrix} \frac{2}{w} & -\frac{2p}{w^2} \\ -\frac{2p}{w^2} & \frac{2p^2}{w^3} \end{vmatrix}$$

In particular, the Hessian is a positive semi-definite matrix, since

$$\frac{2}{w}\frac{2p^2}{w^3} - \left(-\frac{2p}{w^2}\right)\left(-\frac{2p}{w^2}\right) = \frac{4p^2}{w^4} - \frac{4p^2}{w^4} = 0$$

implying that the profit function $\pi(p,w)$ is convex.

(c) Calculate the supply function of the firm, $q(p,w)$, and its demand for inputs, $z(p,w)$.

- Using Hotelling's Lemma we can find the supply function, by differentiating the profit function with respect to p, as follows

$$q(p,\, w) = \frac{\partial \pi(p, w)}{\partial p} = \frac{2p}{w}$$

and the conditional factor demand correspondence can also be found by differentiating the profit function with respect to w, as follows

$$z(p,\, w) = -\frac{\partial \pi(p, w)}{\partial w} = \frac{p^2}{w^2}$$

- Note that both the supply function, $q(p,\, w)$, and the input demand function, $z(p,\, w)$, are increasing in output prices p (more attractive sales) but decreasing in input prices w (i.e., more costly resources).