Homework #4 (10/07/2025)

1. Suppose that a firm owns two plants, each producing the same good. Every plant j's average cost is given by

$$AC_j(q_j) = \alpha + \beta_j q_j \text{ for } q_j \geq 0, \text{ where } j = \{1,2\}$$

where coefficient β_j may differ from plant to plant, i.e., if $\beta_1 > \beta_2$ plant 2 is more efficient than plant 1 since its average costs increase less rapidly in output. Assume that you are asked to determine the cost-minimizing distribution of aggregate output $q = q_1 + q_2$, among the two plants (i.e., for a given aggregate output q, how much q_1 to produce in plant 1 and how much q_2 to produce in plant 2.) For simplicity, consider that aggregate output q satisfies $q < \frac{\alpha}{\max_j |\beta_j|}$. (You will be using this condition in part b.)

   (a) If $\beta_j > 0$ for every plant j, how should output be located among the two plants?

   (b) If $\beta_j < 0$ for every plant j, how should output be located among the two plants?

   (c) If $\beta_j > 0$ for some plants and $\beta_i < 0$ for others?

2. Consider a Cobb-Douglas production function $f : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$, given by $f(z) = 2^{3/4} z_1^{1/4} z_2^{1/4}$, where $z_1 \geqslant 0$ and $z_2 \geqslant 0$ denote inputs in the production process.

   (a) Check if the production function has nonincreasing, nondecreasing, or constant returns to scale.

   (b) Let $w \in \mathbb{R}^2_{++}$ denote the vector of input prices and $p > 0$ the output price. Determine for each output level $q \geqslant 0$ the cost function $c(w,q)$ and the conditional factor demand $z(w,q)$.

   (c) Verify Shephard's lemma.

   (d) Determine the profit function $\pi(p,w)$.

3. Consider a firm whose production function $f(z)$ exhibits constant returns to scale. Show that its cost function can be expressed as $c(w,q) = q \cdot c(w,1)$, i.e., the cost per unit times the number of units produced.

4. Show that, if a production function $f : \mathbb{R}^{L-1} \to \mathbb{R}$ satisfies increasing returns to scale, that is,

$$\text{for every } z \in \mathbb{R}^{L-1} \text{ and for every } t \geqslant 1,\ f(tz) \geqslant t\, f(z)$$

then $f(z)$ also satisfies *increasing average product* property.