Homework #3 (Due on September 25th, 2025)

1. Assume that you have friend, Leo, who is now retired and lives on a fixed income $w > 0$ which does not adjust to inflation. His expenditure function is

$$e(p_1, p_2, u) = (p_1 + p_2)u,$$

where $p_1, p_2 > 0$ denote initial prices. Suppose that prices of goods 1 and 2 increase to p_1' and p_2', respectively.

(a) You want to give him a monetary gift so that he will not be affected by the above price increase. How much money should you give him? That is, find his compensating variation (CV).

- From duality, we know that $e(p_1, p_2, u) = w$, which helps us rewrite the above equation as $w = (p_1 + p_2)u$. Therefore, at the initial price vector, the maximum utility the retiree can obtain is $u = \frac{w}{p_1+p_2}$. In order to ensure that he is not worse off after the price increase, we need to give him an amount of money that covers the difference between the new expense and the old expense, that is,

$$\begin{aligned} CV &= w' - w \\ &= e(p_1', p_2', u) - w \\ &= \underbrace{(p_1' + p_2')u}_{e(p_1', p_2', u)} - w \\ &= (p_1' + p_2')\underbrace{\frac{w}{p_1 + p_2}}_{u} - w \\ &= w\left(\frac{p_1' + p_2'}{p_1 + p_2} - 1\right) \end{aligned}$$

(b) Now find his equivalent variation (EV) from the price change, i.e., the change in income needed at initial prices p_1 and p_2 that would have the same effect on utility as would the change in prices, p_1' and p_2'.

- The equivalent variation is

$$\begin{aligned} EV &= e(p_1, p_2, u') - e(p_1', p_2', u') \\ &= (p_1 + p_2)u' - (p_1' + p_2')u' \\ &= (p_1 + p_2)\underbrace{\frac{w}{p_1' + p_2'}}_{u'} - (p_1' + p_2')\underbrace{\frac{w}{p_1' + p_2'}}_{u'} \\ &= w\left(\frac{p_1 + p_2}{p_1' + p_2'} - 1\right) \end{aligned}$$

(c) Which is larger in this case, CV or EV?

- In this case, since $p_1' + p_2' > p_1 + p_2$, we obtain that the compensating variation is positive $CV > 0$, but the equivalent variation is negative $EV < 0$, entailing that $CV > 0 > EV$.

(d) Find his Walrasian demand for each good.

- *Good 1.* From duality, we know that $e(p_1, p_2, u) = w$, which yields $w = (p_1 + p_2)u$. Solving for u, we find the indirect utility function

$$v(p_1, p_2) = (p_1 + p_2)^{-1}w$$

Then, we can insert this indirect utility function into Roy's identity, as follows, which yields the Walrasian demand for good 1:

$$x_1(p_1, p_2, w) = -\frac{\frac{\partial v(p_1,p_2)}{\partial p_1}}{\frac{\partial v(p_1,p_2)}{\partial w}} = \frac{w(p_1 + p_2)^{-2}}{(p_1 + p_2)^{-1}} = w(p_1 + p_2)^{-1}.$$

- *Good 2.* Following a similar approach, we can find the Walrasian demand for good 2:

$$x_2(p_1, p_2, w) = -\frac{\frac{\partial v(p_1,p_2)}{\partial p_2}}{\frac{\partial v(p_1,p_2)}{\partial w}} = \frac{w(p_1 + p_2)^{-2}}{(p_1 + p_2)^{-1}} = w(p_1 + p_2)^{-1}$$

implying that the consumer purchases the same amount of both goods, $x_1(p_1, p_2, w) = x_2(p_1, p_2, w)$.

(e) Find his utility function. What is this type of utility function called?

- In part (d), we found that the consumer in this exercise, Leo, purchases the same amount of both goods, $x_1(p_1, p_2, w) = x_2(p_1, p_2, w)$, regardless of the price vector and income he faces. This only occurs when the consumer regards both goods as complements, exhibiting a Leontief utility function

$u(x_1, x_2) = A\min\{x_1, x_2\}$, where $A > 0$. As a remark, note that we do not say that his utility function is the general expression of the Leontieff utility function $u(x_1, x_2) = A\min\{ax_1, bx_2\}$, where $A, a, b > 0$ since in such a setting the consumer could purchase different amounts of each good; as long as he keeps the proportion of goods he consumes constant.

2. Chelsea loves chocolate (x) and books (y), and her utility from consuming these two goods can be represented by a with quasilinear utility function $u(x, y) = \rho\sqrt{x} + \tau y$, where $\rho, \tau > 0$.

   (a) Find the Walrassian demand of the individual.

We need to solve the UMP

$$\max_{x,y} \quad u(x, y) = \rho\sqrt{x} + \tau y$$
$$\text{subject to} \quad p_x x + p_y y \le w$$

Taking first-order conditions with respect to x yields

$$\frac{\rho x^{\frac{-1}{2}}}{2} - \lambda p_x \le 0$$

where λ denotes the Lagrange multiplier. Similarly, taking first order conditions with respect to y, we find

$$\tau - \lambda p_y \le 0$$

In the case of interior solutions, the above first order conditions hold with equality. Solving for λ in both of them, we obtain

$$\frac{\rho x^{\frac{-1}{2}}}{2p_x} = \frac{\tau}{p_y}, \quad \text{or} \quad x = \left(\frac{2\tau p_x}{\rho p_y}\right)^{-2}$$

Hence, this consumer's Walrasian demands are

$$x(p, w) = \left(\frac{2\tau p_x}{\rho p_y}\right)^{-2} \quad \text{and} \quad y(p, w) = \frac{w}{p_y} - \frac{p_x}{p_y}\left(\frac{2\tau p_x}{\rho p_y}\right)^{-2} = \frac{w}{p_y} - \frac{p_y}{p_x}\left(\frac{\rho}{2\tau}\right)^2$$

and the indirect utility function becomes

$$\begin{aligned} v(p, w) &= \frac{\rho^2 p_y}{2\tau p_x} + \tau\left[\frac{w}{p_y} - \frac{p_y}{p_x}\left(\frac{\rho}{2\tau}\right)^2\right] \\ &= \frac{\tau w}{p_y} + \frac{1}{2}\left(\frac{\rho^2 p_y}{2\tau p_x}\right) \end{aligned}$$

1. (b) Find the Hicksian demand for goods 1 and 2.

   - Let us solve the following EMP

$$\min_{x,y} \quad p_x x + p_y y$$

$$\text{subject to} \quad \rho\sqrt{x} + \tau y \geq u$$

   Taking first order conditions with respect to x and y, yields

$$p_x \geq \mu \frac{\rho x^{\frac{-1}{2}}}{2}, \text{ and}$$

$$p_y \geq \mu\tau$$

   where μ denotes the Lagrange multiplier of this minimization problem. In the case of interior solutions, the above first order conditions become

$$\begin{cases} p_x = \mu \frac{\rho x^{\frac{-1}{2}}}{2} \\ p_y = \mu\tau \\ \rho\sqrt{x} + \tau y = u \end{cases}$$

   Thus, simultaneously solving for x and y, we find the Hicksian demands for chocolate and books are

$$h_x(p,u) = \left(\frac{2\tau p_x}{\rho p_y}\right)^{-2} \qquad h_2(p,u) = \frac{u}{\tau} - \left(\frac{\tau p_x^{\frac{1}{2}}}{\rho^{\frac{1}{2}}{}^{\frac{1}{2}} p_y^{\frac{1}{2}}}\right)^{-2} = \frac{u}{\tau} - \frac{2p_y}{p_x}\left(\frac{\rho}{\tau}\right)^2 \left(\frac{\tau p_x^{\frac{1}{2}}}{\rho^{\frac{1}{2}}{}^{\frac{1}{2}} p_y^{\frac{1}{2}}}\right)^{-2}$$

and the expenditure function becomes

$$\begin{aligned} e(p,u) &= \frac{\rho^2 p_y}{2\tau p_x} + \tau\left(\frac{u}{\tau} - \frac{\rho^2 p_y}{2\tau^2 p_x}\right) \\ &= u \end{aligned}$$

c) Assume that Chelsea's wealth is $w = \$500$, and prices are $p_1 = p_2 = \$15$. For simplicity, consider parameters $\rho = 1, \tau = \frac{1}{2}$. Find the AV, CV and EV.

For these parameters, the Walrasian demand of both goods are strictly positive (check as a practice). Hence, the indirect utility function is

$$
\begin{aligned}
v(p,w) &= \frac{\tau w}{p_y} + \frac{1}{2}\left(\frac{\rho^2 p_y}{2\tau p_x}\right) \\
&= \frac{\frac{1}{2}\times 500}{15} + \left(\frac{1^2\times 15}{2\times\frac{1}{2}\times 15}\right) \\
&= \frac{50}{3} + \frac{1}{2} \\
&= \frac{103}{6} \\
&\approx 17.1667
\end{aligned}
$$

- *Area variation.* Hence, if the price of good 1 decreases by 50%, i.e., from $p_x = 15$ to $p'_x = 7.5$, the area variation is

$$
\begin{aligned}
AV &= \int_{7.5}^{15} x(p,w)dp_x \\
&= \int_{7.5}^{15} \left(\frac{2\times p_x}{\rho\times p_y}\tau\right)^{-2} dp_x \\
&= \int_{7.5}^{15} \left(\frac{2}{1\times 15}\times\frac{1}{2}\right)^{-2} p_x^{-2} dp_x \\
&= \left[-15^2\frac{1}{p_x}\right]_{7.5}^{15} \\
&= 15
\end{aligned}
$$

- *Compensating Variation.* Let us now examine the compensating variation associated with this price decrease. First, recall that the Hicksian demands (found in part b) are

$$
h_x(p,u) = \left(\frac{\rho p_y}{2\tau p_x}\right)^2
$$

$$
h_y(p,u) = \frac{u}{\tau} - \frac{\rho^2 p_y}{2\tau^2 p_x}
$$

Hence, the compensating variation associated to a 50% decrease in the price of good 1 is

$$\begin{aligned} CV &= \int_{7.5}^{15} h_x(p, u^o) dp_x \\ &= \int_{7.5}^{15} \left(\frac{\rho p_y}{2\tau p_x} \right)^2 dp_x \\ &= \int_{7.5}^{15} \left(\frac{1 \times 15}{2 \times \frac{1}{2} p_x} \right)^2 dp_x \\ &= \left[-15^2 p_x^{-1} \right]_{7.5}^{15} \\ &= 15 \end{aligned}$$

- *Equivalent Variation.* Let us finally identify the equivalent variation of this price change. In order to do this, we first need to evaluate the indirect utility function $v(p; w)$ at the final prices $p_x = 7.5$ and $p_y = 15$, which yields u^1. Plugging this utility level u^1 on the Hicksian demand for good 1 (also evaluated at final prices $p_x = 7.5$ and $p_y = 15$), we obtain the equivalent variation. Notice that here $h_x(p, u^1)$ does not contain u^1 in the equation, we can derive EV as

$$\begin{aligned} EV &= \int_{7.5}^{15} h_x(p, u^1) dp_x \\ &= \int_{7.5}^{15} \left(\frac{\rho p_y}{2\tau p_x} \right)^2 dp_x \\ &= \int_{7.5}^{15} \left(\frac{1 \times 15}{2 \times \frac{1}{2} p_x} \right)^2 dp_x \\ &= \left[-15^2 p_x^{-1} \right]_{7.5}^{15} \\ &= 15 \end{aligned}$$

Therefore, since the utility function is quasilinear, we confirmed that all welfare measures coincide, i.e., $AV = CV = EV$.

3. Consider an individual with utility function $u(q_1, q_2) = q_1^2 + q_2 - 1$, where q_1 (q_2) denotes the units of good 1 (good 2, respectively) that this individual consumes. His income level is denoted by $w \in \mathbb{R}_+$, and prices are both strictly positive, i.e., $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2_{++}$.

   (a) Determine this individual's Walrasian demand, and his associated indirect utility function.

   - Before starting the exercise, note that this individual's indifference curves are strictly concave in the $\{q_1, q_2\}$ space. Indeed, solving this individual's utility function for q_2, we obtain $q_2(q_1, u) = -q_1^2 + u + 1$. Hence, their first and second-order derivatives are

$$q_2'(q_1, u) = -2q_1 < 0 \text{ and } q_2''(q_1, u) = -2 < 0$$

i.e., indifference curves have a negative slope that becomes more negative as q_1 increases (bowed away from the origin). Furthermore, its vertical intercept is $(0, u+1)$ and its horizontal intercept is $(\sqrt{u+1}, 0)$, as depicted in figure 1.

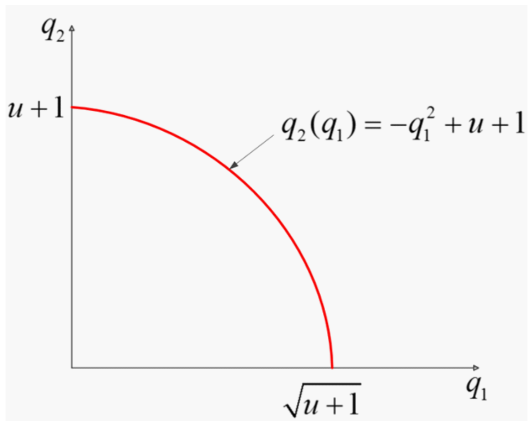

Figure 1. Indifference curve for $u(q_1, q_2) = q_1^2 + q_2 - 1$.

- *Walrasian demands.* Such a concavity helps us anticipate that the consumer's Walrasian demands will be at corner solution, i.e., $\left(\frac{w}{p_1}, 0\right)$ or $\left(0, \frac{w}{p_2}\right)$. If we compare the utility levels associated to each of these bundles, we obtain that the consumer prefers $\left(\frac{w}{p_1}, 0\right)$ to $\left(0, \frac{w}{p_2}\right)$ if and only if $u\left(\frac{w}{p_1}, 0\right) > u\left(0, \frac{w}{p_2}\right)$, which implies $\frac{w^2}{p_1^2} - 1 > \frac{w}{p_2} - 1$, i.e., $w > \frac{p_1^2}{p_2}$. We can hence express the Walrasian demands as

$$\left(q_1^W(\mathbf{p}, w), q_2^W(\mathbf{p}, w)\right) = \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if } w > \frac{p_1^2}{p_2}, \text{ and} \\ \left(0, \frac{w}{p_2}\right) & \text{if } w \leq \frac{p_1^2}{p_2} \end{cases} \tag{1}$$

Intuitively, if $w > \frac{p_1^2}{p_2}$ (which we can alternatively express as $p_2 > \frac{p_1^2}{w}$) good 2 is expensive (relative to the price of good 1 and the individual's wealth level), leading the consumer to buy good 1 alone. Otherwise, the consumer only buys good 2. Given (1), the indirect utility function becomes

$$v_1(\mathbf{p}, w) = \begin{cases} \frac{w^2}{p_1^2} - 1 \text{ if } w > \frac{p_1^2}{p_2}, \text{ and} \\ \frac{w}{p_2} - 1 \text{ if } w \leq \frac{p_1^2}{p_2} \end{cases} \tag{2}$$

(b) Determine this individual's Hicksian demand, $h_1(\mathbf{p}, u)$ and $h_2(\mathbf{p}, u)$, and his associated expenditure function, $e(\mathbf{p}, u)$.

- Because of concavity on the consumer's indifference curves, hicksian demands must also be at corner solution. Hence, when $q_1 = 0$ we must have that $q_2 = u + 1$; while if $q_2 = 0$ we must have that $q_1 = \sqrt{u+1}$. We therefore have two corner solutions: $(0, u+1)$ or $\left(\sqrt{u+1}, 0\right)$. Since they both lie on the same indifference curve, the consumer chooses the less expensive of the two. In particular, the cost of buying bundle $(0, u+1)$ is larger than that of bundle $\left(\sqrt{u+1}, 0\right)$ if

$$p_1 0 + p_2 (u+1) > p_1\sqrt{u+1} + p_2 0 \Longleftrightarrow u > \frac{p_1^2}{p_2^2} - 1$$

We hence obtain the Hicksian demands

$$(h_1(\mathbf{p}, u), h_2(\mathbf{p}, u)) = \begin{cases} \left(\sqrt{u+1}, 0\right) & \text{if } u > \frac{p_1^2}{p_2^2} - 1, \text{ and} \\ (0, u+1) & \text{if } u \leq \frac{p_1^2}{p_2^2} - 1 \end{cases} \tag{3}$$

As a consequence, the expenditure function is

$$e(\mathbf{p}, u) = \begin{cases} p_1\sqrt{u+1} & \text{if } u > \frac{p_1^2}{p_2^2} - 1, \text{ and} \\ p_2 (u+1) & \text{if } u \leq \frac{p_1^2}{p_2^2} - 1 \end{cases}$$

Consider now that this individual's income level is $w = 6$, and the initial vector of market prices is $\mathbf{p^0} = (4, 3)$. If both prices increase by 50%, determine:

c. The compensating variation of this price increase. Interpret.

- We now have all the elements to provide a monetary evaluation of the welfare change associated to the price increase. The initial price vector and wealth $(\mathbf{p^0}, w) = (4, 3, 6)$ satisfy condition $w > \frac{p_1^2}{p_2}$, i.e., $6 > \frac{4^2}{3} = 5.33$. Therefore, Walrasian demands in this setting are $q_1^W(\mathbf{p}, w) = \frac{w}{p_1} = \frac{6}{4} = \frac{3}{2}$ and $q_2^W(\mathbf{p}, w) = 0$. When prices increase to $\mathbf{p^1} = \left(6, \frac{9}{2}\right)$, the opposite condition holds, i.e., $w < \frac{p_1^2}{p_2}$, since $6 < \frac{6^2}{9/2} = 8$, implying that Walrasian demands after the price increase are $q_1^W(\mathbf{p}, w) = 0$ and $q_2^W(\mathbf{p}, w) = \frac{w}{p_2} = \frac{6}{9/2} = \frac{4}{3}$. We hence need to evaluate the welfare change that arises when "jumping" from the corner bundle $\left(\frac{3}{2},0\right)$ to the other corner bundle $\left(0,\frac{4}{3}\right)$; as depicted in figure 2.

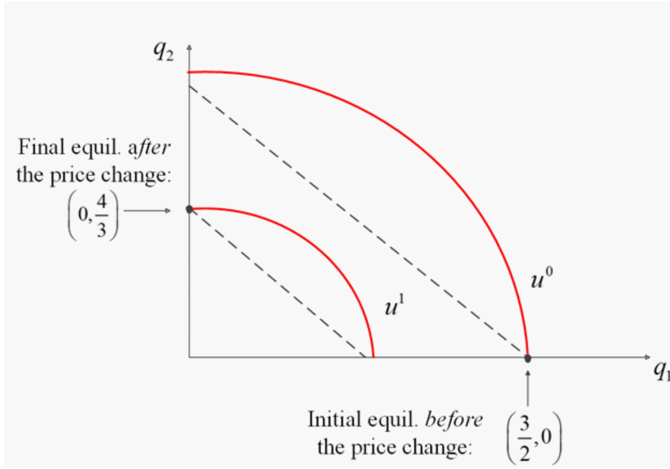

Figure 2. Walrasian demand before and after the price increase.

The expenditure function at the initial price level is $e(\mathbf{p^0},u^0)=p_1^0\sqrt{u^0+1}$, where $u^0=\frac{w^2}{p_1^2}-1=\frac{5}{4}$. Therefore, $e(\mathbf{p^0},\frac{5}{4})=4\sqrt{\frac{5}{4}+1}=6=w$. Similarly, at the final price level, the expenditure function becomes $e(\mathbf{p^1},\frac{5}{4})=6\sqrt{\frac{5}{4}+1}=9$.[1] As a consequence, the compensating variation (CV) is

$$CV=e\left(\mathbf{p}^1,\frac{5}{4}\right)-e\left(\mathbf{p}^0,\frac{5}{4}\right)=9-6=\$3$$

Intuitively, the income level should be increased in \$3, to $w+CV=\$9$, in order to guarantee that the consumer can still reach his initial utility level of u^0 at the new (higher) price level, i.e., we need to increase his income level by 50%.

d. The change in consumer surplus associated to this price increase. Interpret.

- The change in consumer surplus, also referred as Area Variation (AV), that

[1] Note that the expression of $e(\mathbf{p},u)$ depends on whether condition $u>\frac{p_1^2}{p_2^2}-1$ or $u\leq\frac{p_1^2}{p_2^2}-1$ holds. This condition is, however, unaffected by a proportional increase in the price of both goods. Therefore, the cheapest bundle that reaches utility u^0 is $\left(\frac{3}{2},0\right)$ before and after the price increase. Indeed, $p_1^1\sqrt{u^0+1}=6\sqrt{\frac{5}{4}+1}<\frac{9}{2}\left(\frac{5}{4}+1\right)=p_2^1\left(u^0+1\right)$, implying that $\left(\frac{3}{2},0\right)$ is still cheaper than $\left(0,\frac{4}{3}\right)$ at the new prices.

arises from the "jump" from one corner solution to another, is given by

$$
\begin{aligned}
AV &= \overbrace{\int_4^{3\sqrt{2}} \frac{w}{p_1} dp_1 + \int_{3\sqrt{2}}^{6} 0 dp_1}^{p_2^0=3} + \overbrace{\int_3^{9/2} \frac{w}{p_2} dp_2}^{p_1^1=6} \\
&= 6 \ln \left(\frac{9\sqrt{2}}{8} \right) \simeq 2.78
\end{aligned}
$$

Explanation of three elements in AV: We first increase p_1 in the first two terms, and p_2 in the third term. Here is a more detailed explanation (Figures 3a and 3b illustrate the Walrasian demand for good 1 and 2, respectively):

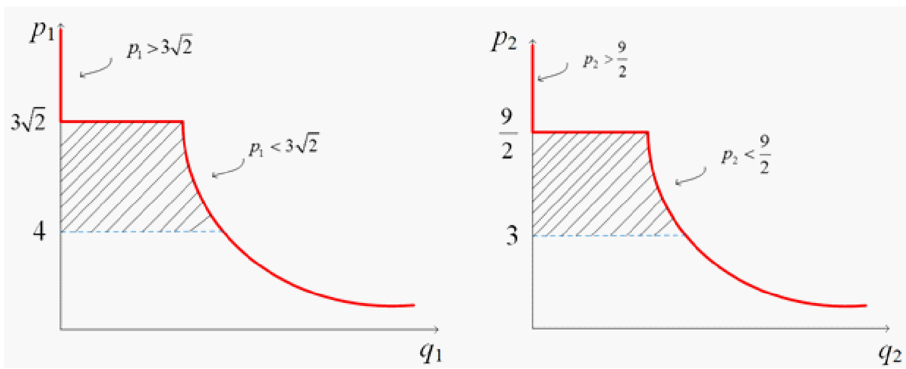

Figure 3a. Good 1. Figure 3b. Good 2.

- Given an income level $w = 6$ and a price for good 2 of $p_2 = 3$, condition $w > \frac{p_1^2}{p_2}$ holds if $6 > \frac{p_1^2}{3}$, i.e., for all $p_1 < 3\sqrt{2} \simeq 4.24$. Hence if $p_1 < 3\sqrt{2}$, the Walrasian demand for good 1 is $q_1^W(\mathbf{p}, w) = \frac{w}{p_1}$.
- If, instead, $p_1 \geq 3\sqrt{2}$ holds, the Walrasian demand for good 1 collapses to zero, i.e., $q_1^W(\mathbf{p}, w) = 0$.
- Once p_1 is at the final price level $p_1 = \$6$, we can increase p_2. In particular, we use condition $w \gtrless \frac{p_1^2}{p_2}$, to determine that $6 > \frac{6^2}{p_2}$ for all $p_2 \in \left[3, \frac{9}{2}\right]$, implying that the demand for good 2 is $q_2^W(\mathbf{p}, w) = \frac{w}{p_2}$.

4. Consider a firm with production function $q = \sqrt{z}$, using one input (e.g., labor) to produce one type of output. The price of every unit of input is $w = 8$, and the price of every unit of output is $p > 0$.

   (a) Set up the firm's profit-maximization problem, and solve for its unconditional factor demand $z(8, p)$.

- The firm chooses the units of input z to solve

$$\max_{z \geq 0} p\sqrt{z} - 8z$$

where the first term indicates total revenue, whereas the second reflects total costs. Taking first-order condition with respect to z, we obtain

$$p\frac{1}{2}z^{-1/2} - 8 \leq 0.$$

In the case of interior solutions, we can solve for z to find the unconditional factor demand

$$z(8, p) = \frac{p^2}{256}.$$

Hence, total output is $q = \sqrt{\frac{p^2}{256}} = \frac{p}{16}$ units.

(b) Evaluate the profit function at the unconditional factor demand $z(8, p)$. Test for convexity of the profit function in output price p.

- Inserting $z(8, p) = \frac{p^2}{256}$ into the firm's objective function, we obtain

$$\pi(p) = p\sqrt{z(8, p)} - 8z(8, p) = \frac{1}{32}p\,(2p - p) = \frac{p^2}{32},$$

which is convex in output price p.

(c) Let us now illustrate convexity in output prices by using an alternative approach: (1) evaluate the profit function you found in part (b) at prices $p = 6$, and at $p = 12$. Then, find their convex combination $\alpha\pi(6) + (1 - \alpha)\,\pi(12)$ where $\alpha \in [0, 1]$; (2) evaluate the profit function at the convex combination of the above output prices, that is, $\pi(\alpha 6 + (1 - \alpha)\,12)$. Last, show that the profit function you found in step (1) lies weakly above that found in step (2) for all values of α, that is,

$$\alpha\pi(6) + (1 - \alpha)\,\pi(12) \geq \pi(\alpha 6 + (1 - \alpha)\,12).$$

- Evaluating the output function at those two output prices, we obtain $\pi(6) = \frac{9}{8}$ and $\pi(12) = \frac{9}{2}$. Hence, their convex combination is

$$\alpha\pi(6) + (1 - \alpha)\,\pi(12) = \alpha\frac{9}{8} + (1 - \alpha)\,\frac{9}{2} = \frac{9}{8}(4 - 3\alpha).$$

If, instead, we evaluate the profit function at an output price $p = \alpha 6 +$

$(1-\alpha)\,12$, we obtain

$$\pi\,(\alpha 6 + (1-\alpha)\,12) = \frac{9}{8}(4 - 4\alpha + \alpha^2)$$

Substracting $[\alpha\pi(6) + (1-\alpha)\,\pi(12)] - \pi(\alpha 6 + (1-\alpha)\,12)$, we find

$$\frac{9}{8}(4 - 3\alpha) - \frac{9}{8}(4 - 4\alpha + \alpha^2) = \frac{9}{8}\alpha\,(1-\alpha) > 0.$$

which is positive since $\alpha \in [0,1]$.