Applications of Game Theory to Environmental Problems

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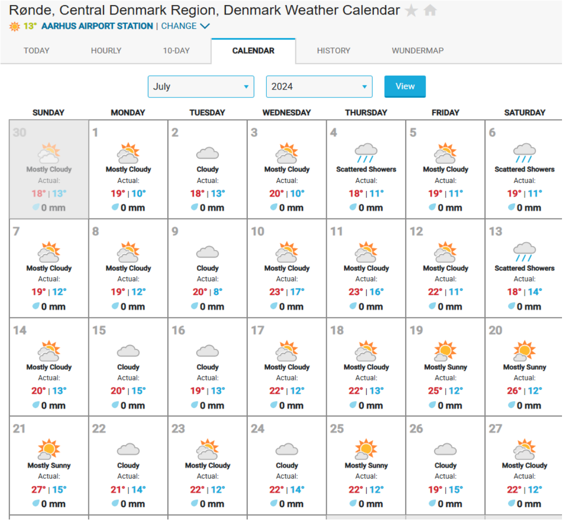
July 2025

Introduction

- Decision in the dark!

- Examples...

Introduction



Rønde, Central Denmark Region, Denmark Weather Calendar

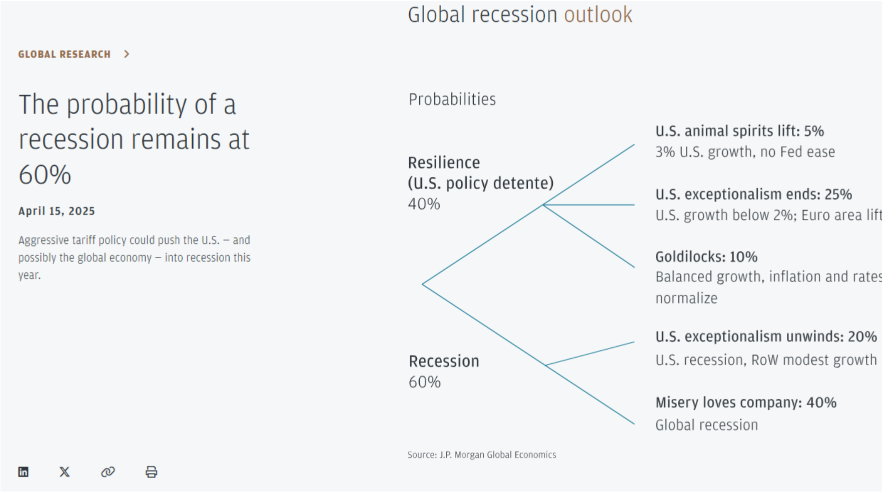
13° AARHUS AIRPORT STATION | CHANGE

July

2024

SUNDAY	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY	SATURDAY
30 Mostly Cloudy Actual: 18° | 13° 0 mm	1 Mostly Cloudy Actual: 19° | 10° 0 mm	2 Cloudy Actual: 18° | 13° 0 mm	3 Mostly Cloudy Actual: 20° | 10° 0 mm	4 Scattered Showers Actual: 18° | 11° 0 mm	5 Mostly Cloudy Actual: 19° | 11° 0 mm	6 Scattered Showers Actual: 19° | 11° 0 mm
7 Mostly Cloudy Actual: 19° | 12° 0 mm	8 Mostly Cloudy Actual: 19° | 12° 0 mm	9 Cloudy Actual: 20° | 8° 0 mm	10 Mostly Cloudy Actual: 23° | 17° 0 mm	11 Mostly Cloudy Actual: 23° | 16° 0 mm	12 Mostly Cloudy Actual: 22° | 11° 0 mm	13 Scattered Showers Actual: 18° | 14° 0 mm
14 Mostly Cloudy Actual: 20° | 13° 0 mm	15 Cloudy Actual: 20° | 15° 0 mm	16 Cloudy Actual: 19° | 13° 0 mm	17 Mostly Cloudy Actual: 22° | 12° 0 mm	18 Mostly Cloudy Actual: 22° | 13° 0 mm	19 Mostly Sunny Actual: 25° | 12° 0 mm	20 Mostly Sunny Actual: 26° | 12° 0 mm
21 Mostly Sunny Actual: 27° | 15° 0 mm	22 Cloudy Actual: 21° | 14° 0 mm	23 Mostly Cloudy Actual: 22° | 12° 0 mm	24 Cloudy Actual: 22° | 14° 0 mm	25 Mostly Sunny Actual: 22° | 12° 0 mm	26 Cloudy Actual: 19° | 15° 0 mm	27 Mostly Cloudy Actual: 22° | 12° 0 mm

Introduction



GLOBAL RESEARCH >

The probability of a recession remains at 60%

April 15, 2025

Aggressive tariff policy could push the U.S. – and possibly the global economy – into recession this year.

Global recession outlook

Probabilities

Resilience (U.S. policy detente)
40%

U.S. animal spirits lift: 5%
3% U.S. growth, no Fed ease

U.S. exceptionalism ends: 25%
U.S. growth below 2%; Euro area lift

Goldilocks: 10%
Balanced growth, inflation and rates normalize

Recession
60%

U.S. exceptionalism unwinds: 20%
U.S. recession, RoW modest growth

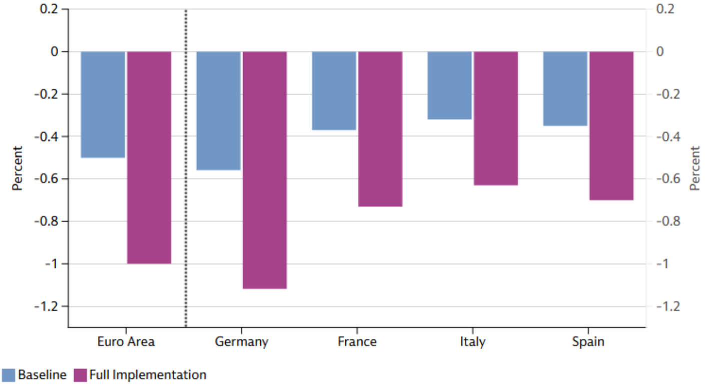
Misery loves company: 40%
Global recession

Source: J.P. Morgan Global Economics

Introduction

Goldman Sachs Research expects a 0.5% GDP hit from trade tensions

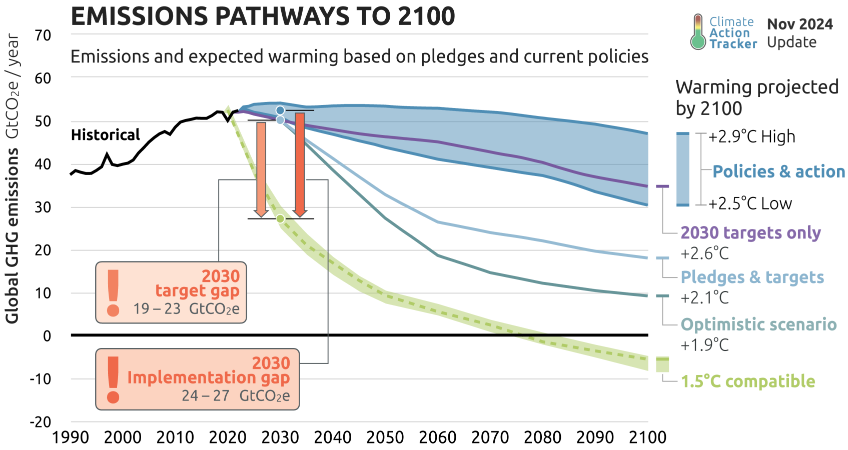
Impact of US tariffs on GDP Level



Source: Goldman Sachs Research, Haver Analytics
Full implementation assumes a 10% tariff on all US imports (including from Europe) and baseline assumes a more limited set of tariffs on Europe, including on autos-related imports, and tariffs on China.

Goldman Sachs

Introduction



Looking back...

- So far we have been able to find the NE of a relatively large class of games with complete information:
  - Games with two or several ($n > 2$) players.
  - Games where players select among discrete or continuous actions.
- But, can we assure that all complete information games where players select their actions simultaneously have a NE?
  - We couldn't find a NE for the matching pennies game!! (Next slide)
  - We will be able to claim existence of a NE if we allow players to randomize their actions.

Remembering the "matching pennies" game...

- Recall that this was an example of an anti-coordination game:

		P_2	
		Head	Tail
P_1	Head	$\underline{1}$, -1	-1, $\underline{1}$
	Tail	-1, $\underline{1}$	$\underline{1}$, -1

  Indeed, there is no strategy pair in which players select a particular action 100% of the times.

- We need to allow players to randomize their choices.

Another example

- Here we have another example of an anti-coordination game with no psNE:

Surprise!

		Drug Dealer	
		Street Corner	Park
Police Officer	Street Corner	$\underline{80}, 20$	$0, \underline{100}$
	Park	$10, \underline{90}$	$\underline{60}, 40$

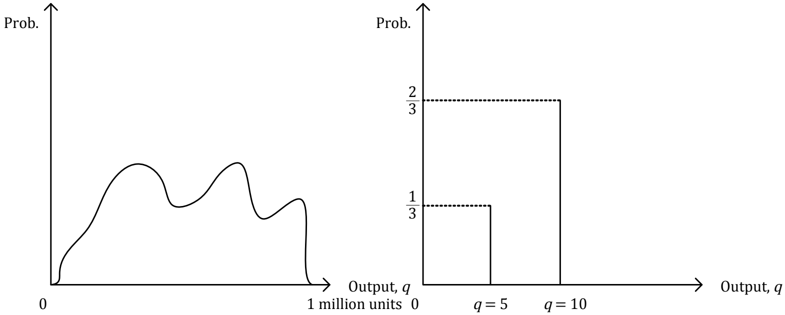
- We need to allow players randomize their choices (i.e., to play mixed strategies).

Mixed strategy Nash equilibrium

- Harrington: Chapter 7, Watson: Chapter 11.
- First, note that if a player plays more than one strategy with strictly positive probability, then he must be indifferent between the strategies he plays with strictly positive probability.
- **Notation**: "non-degenerate" mixed strategies denotes a set of strategies that a player plays with strictly positive probability.
    - Whereas "degenerate" mixed strategy is just a pure strategy (because of degenerate probability distribution concentrates all its probability weight at a single point).

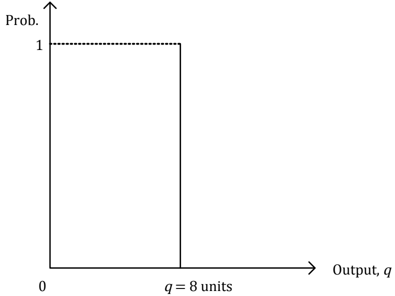
Degenerate Probability Distributions

- Example of non-degenerate probability distributions



Degenerate Probability Distributions

- Example of a degenerate probability distribution



- The player (e.g., firm) puts all probability weight (100%) on only one of its possible actions: $q = 8$.

- **Definition of msNE:**
  - Consider a strategy profile $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ where σ_i is a mixed strategy for player i. σ is a msNE if and only if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}) \text{ for all } s_i' \in S_i \text{ and for all } i$$

  - That is, σ_i is a best response of player i to the strategy profile σ_{-i} of the other $N-1$ players, $\sigma_i = BR_i(\sigma_{-i})$.

- Notice that we wrote $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\mathbf{s}'_i, \sigma_{-i})$ instead of $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\boldsymbol{\sigma}'_i, \sigma_{-i})$.
- **Why?** If a player was using σ'_i, then he would indifferent between all pure strategies to which σ'_i puts a positive probability, for example $\hat{s}_i$ and $\check{s}_i$.
  - That is why it suffices to check that no player has a profitable pure-strategy deviation.

Example 1:Matching pennies

- **Matching pennies**

			Player 2	
			q	$1-q$
			Heads	Tails
Player 1	p	Heads	$1, -1$	$-1, 1$
	$1-p$	Tails	$-1, 1$	$1, -1$

- **Two alternative interpretations of players' randomization:**
  - If player 1 is using a mixed strategy, it must be that he indifferent between Heads and Tails
  - Alternatively, if player 1 is indifferent between Heads and Tails, it must be that player 2 mixes with such probability q such that player 1 is made indifferent between Heads and Tails:

$$EU_1(H) = EU_1(T) \iff 1q + (1-q)(-1) = (-1)q + 1(1-q)$$

Matching pennies

- **Matching pennies** (example of a normal form game with no psNE):

			Player 2	
			q	$1-q$
			Heads	Tails
Player 1	p	Heads	$1, -1$	$-1, 1$
	$1-p$	Tails	$-1, 1$	$1, -1$

- Solving for the EU comparison, we obtain

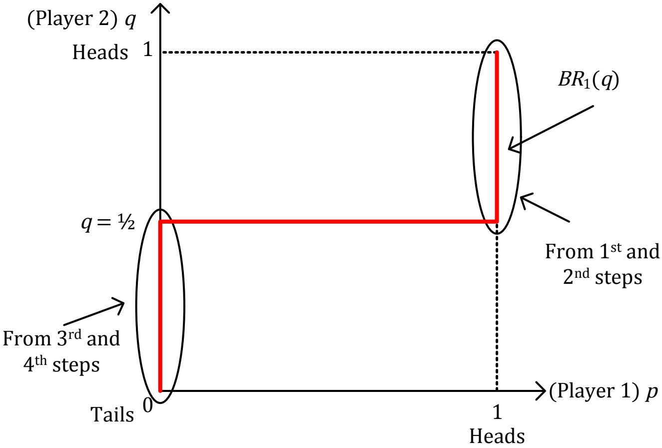
$$EU_1(H) = EU_1(T) \iff 1q + (1-q)(-1) = (-1)q + 1(1-q)$$

$$q = \frac{1}{2} \longrightarrow \text{Graphical Interpretation}$$

Matching pennies

- How to interpret this cutoff of $q = \frac{1}{2}$ graphically?
    1. We know that if $q > \frac{1}{2}$, then player 2 is very likely playing Heads. Then, player 1 prefers to play Heads as well ($p = 1$).
        - Alternatively, note that $q > \frac{1}{2}$ implies $EU_1(H) > EU_1(T)$.
    2. Go to the figure on the next slide, and draw $p = 1$ for every $q > \frac{1}{2}$.
    3. If $q < \frac{1}{2}$, player 2 is likely playing Tails. Then, player 1 prefers to play Tails as well ($p = 0$).
    4. Graphically, draw $p = 0$ for every $q < \frac{1}{2}$.

Matching pennies



Matching pennies

- Similarly, if player 2 is using a mixed strategy, it must be that he is indifferent between Heads and Tails:

$$EU_2(H) = EU_2(T)$$

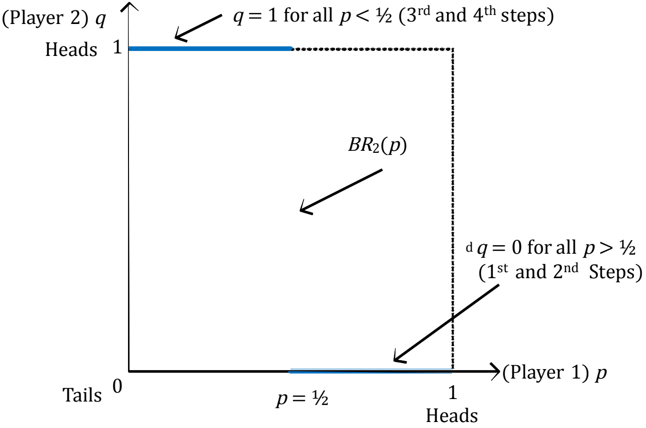
$$(-1)p + 1(1-p) = 1p + (-1)(1-p) \iff p = \frac{1}{2}$$

- (See figure after next slide)

Matching pennies

- Player 2
  1. We know that if $p > \frac{1}{2}$, player 1 is likely playing heads. Then player 2 wants to play tails instead, i.e., $q = 0$.
  2. Go to the figure on the next slide, and draw $q = 0$ for all $p > \frac{1}{2}$.
  3. If $p < \frac{1}{2}$, player 1 is likely playing tails. Then player 2 wants to play heads, i.e., $q = 1$.
  4. Graphically, draw $q = 1$ for all $p < \frac{1}{2}$.

Matching pennies



$q = 1$ for all $p <$ ½ (3rd and 4th steps)

$BR_2(p)$

d $q = 0$ for all $p >$ ½ (1st and 2nd Steps)

Matching pennies

- We can represent these BRFs as follows:
    - **Player 1**

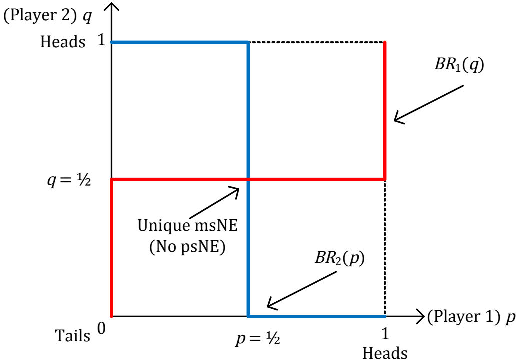
$$BR_1(q) = \begin{cases} \text{Heads if } q > \frac{1}{2} \\ \{\text{Heads, Tails}\} \text{ if } q = \frac{1}{2} \\ \text{Tails if } q < \frac{1}{2} \end{cases}$$

        - Player 1 is indifferent between Heads and Tails when q is exactly $q = \frac{1}{2}$
    - **Player 2**

$$BR_2(p) = \begin{cases} \text{Tails if } p > \frac{1}{2} \\ \{\text{Heads, Tails}\} \text{ if } p = \frac{1}{2} \\ \text{Heads if } p < \frac{1}{2} \end{cases}$$

        - Player 2 is indifferent between Heads and Tails when p is exactly $p = \frac{1}{2}$

Matching pennies



- **Player 1**: When $q > \frac{1}{2}$, Player 1 prefers to play Heads $(p = 1)$; otherwise, Tails.
- **Player 2**: When $p > \frac{1}{2}$, Player 2 prefers to play Tails $(q = 0)$; otherwise, Heads.

Matching pennies

- Therefore, the msNE of this game can be represented as

  $$\left\{\left(\frac{1}{2}H, \frac{1}{2}T\right), \left(\frac{1}{2}H, \frac{1}{2}T\right)\right\}$$

  where the first parenthesis refers to player 1(row player), and the player 2(column player).

Battle of the sexes

2. **Battle of the sexes** (example of a normal form game with 2 psNE already!):

			Wife	
			q	1 - q
			Football	Opera
Husband	p	Football	$\underline{3}, \underline{1}$	0, 0
	1 - p	Opera	0, 0	$\underline{1}, \underline{3}$

If the Husband is using a mixed strategy, it must be that he indifferent between Football and Opera:

$$\begin{aligned} EU_1(F) &= EU_1(O) \\ 3q + 0(1-q) &= 0q + 1(1-q) \\ 3q &= 1 - q \\ 4q &= 1 \implies q = \frac{1}{4} \end{aligned}$$

Battle of the sexes

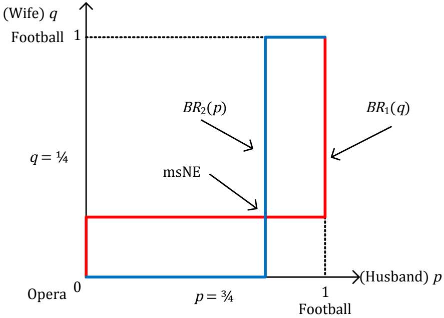
Similarly, if the Wife is using a mixed strategy, it must be that she is indifferent between Football and Opera:

$$\begin{aligned} EU_2(F) &= EU_2(O) \\ & \qquad\qquad \left.\vphantom{\begin{matrix}a\\b\\c\\d\end{matrix}}\right\} \text{Practice!} \\ p &= \frac{3}{4} \end{aligned}$$

Therefore, the msNE of this game can be represented as

$$\text{msNE} = \left\{ \underbrace{\left(\frac{3}{4}F, \frac{1}{4}O\right)}_{\text{Husband}}, \underbrace{\left(\frac{1}{4}F, \frac{3}{4}O\right)}_{\text{Wife}} \right\}$$

Battle of the sexes



- **Husband:** When $q > \frac{1}{4}$, he prefers to go to the Football game $(p = 1)$; otherwise, the Opera.
- **Wife:** When $p > \frac{3}{4}$, she prefers to go to the Football game $(q = 1)$; otherwise, the Opera.

Battle of the sexes

- Best Responses for Battle of the Sexes are hence:
  - **Player 1 (Husband)**

$$BR_1(q) = \begin{cases} \text{Football if } q > \frac{1}{4} \\ \{\text{Football, Opera}\} \text{ if } q = \frac{1}{4} \\ \text{Opera if } q < \frac{1}{4} \end{cases}$$

  - **Player 2 (Wife)**

$$BR_2(p) = \begin{cases} \text{Football if } p > \frac{3}{4} \\ \{\text{Football, Opera}\} \text{ if } p = \frac{3}{4} \\ \text{Opera if } p < \frac{3}{4} \end{cases}$$

Battle of the sexes

- Note the differences in the cutoffs: They reveal each player's preferences.
  - **Husband:** "I will go to the football game as long as there is a slim probability that my wife will be there."
  - **Wife:** "I will only go to the football game if there is more than a 75% chance my husband will be there."

Prisoner's Dilemma

3. **Prisoner's Dilemma** (One psNE, but are there any msNE?):

			Player 2	
			q	1 - *q*
			Confess	Not Confess
Player 1	*p*	Confess	$\underline{-5}, \underline{-5}$	$\underline{0}, -15$
	1 - *p*	Not Confess	$-15, \underline{0}$	-1, -1

If the first player is using a mixed strategy, it must be that he indifferent between Confess and Not Confess:

$$\begin{aligned} EU_1(C) &= EU_1(NC) \\ -5q + 0(1-q) &= -15q + (-1)(1-q) \\ -5q &= -15q - 1 + q \\ 9q &= -1 \implies q = -\frac{1}{9}? \end{aligned}$$

Prisoner's Dilemma

- Similarly, if player 2 is using a mixed strategy, it must be that she is indifferent between Confess and Not Confess:

$$\begin{aligned} EU_2(C) &= EU_2(NC) \\ -5p + 0(1-p) &= -15p + (-1)(1-p) \\ -5p &= -15p - 1 + p \\ 9p &= -1 \implies p = -\frac{1}{9} \end{aligned}$$

- Hence, such msNE would not assign any positive weight to strategies that are strictly dominated.
  - Some textbooks refer to this result by saying that "the support of the msNE is positive only for strategies that are not strictly dominated."

Tennis game (msNE with three available strategies)

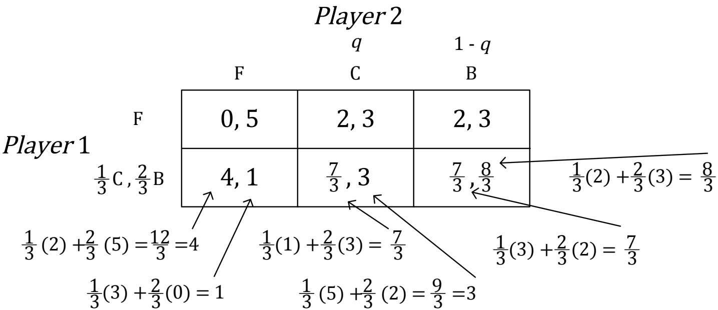
4. **Tennis game** (No psNE, but how do we operate with 3 strategies?):

				Player 2	
				q	1 - q
			F	C	B
		F	0, $\underline{5}$	2, 3	2, 3
Player 1	p	C	2, 3	1, $\underline{5}$	$\underline{3}$, 2
	1 - p	B	$\underline{5}$, 0	$\underline{3}$, 2	2, $\underline{3}$

- Remember this game? We used it as an example of how to delete an strategy that was strictly dominated by the combination of two strategies of that player.
  - Let's do it again.

Tennis game (msNE with three available strategies)

- F is strictly dominated for Player 1:



Player 2

			q	$1 - q$
		F	C	B
Player 1	F	0, 5	2, 3	2, 3
	$\frac{1}{3}$C , $\frac{2}{3}$B	4, 1	$\frac{7}{3}$, 3	$\frac{7}{3}$, $\frac{8}{3}$

$\frac{1}{3}(2) + \frac{2}{3}(3) = \frac{8}{3}$

$\frac{1}{3}(2) + \frac{2}{3}(5) = \frac{12}{3} = 4$

$\frac{1}{3}(1) + \frac{2}{3}(3) = \frac{7}{3}$

$\frac{1}{3}(3) + \frac{2}{3}(2) = \frac{7}{3}$

$\frac{1}{3}(3) + \frac{2}{3}(0) = 1$

$\frac{1}{3}(5) + \frac{2}{3}(2) = \frac{9}{3} = 3$

- We can hence rule out F from Player 1 because it is strictly dominated by $(\frac{1}{3}C, \frac{2}{3}B)$.

Tennis game (msNE with three available strategies)

- After deleting *F* from Player 1's available actions, we are left with:

			Player 2	
		~~F~~	C	B
Player 1	C	~~2, 3~~	1, 5	3, 2
	B	~~5, 0~~	3,2	2, 3

- Where we can rule out *F* from Player 2 because of being strictly dominated by *C*.

Tennis game (msNE with three available strategies)

- Once strategy F has been deleted for both players, we are left with:

Player 2

			q	$1 - q$
			C	B
Player 1	p	C	1, $\underline{5}$	$\underline{3}$, 2
	$1 - p$	B	$\underline{3}$, 2	2, $\underline{3}$

- But we cannot identify any psNE, Let's check for msNE:
- If the first player is using a mixed strategy, it must be that he indifferent between C and B:

$$\left.\begin{aligned} EU_1(C) &= EU_1(B) \quad \ldots. \\ q &= \frac{1}{3} \end{aligned}\right\} \text{Practice!}$$

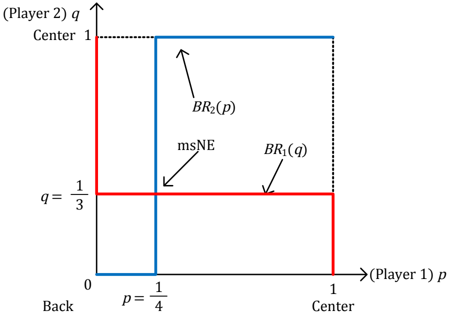
Tennis game (msNE with three available strategies)

- Similarly, if player 2 is using a mixed strategy, it must be that she is indifferent between C and B:

$$\begin{aligned} EU_2(C) &= EU_2(NC) \quad \ldots \\ & \qquad\qquad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Practice!} \\ p &= \frac{1}{4} \end{aligned}$$

- (See figure on next slide)

Tennis game (msNE with three available strategies)



- **Player 1:** If $q > \frac{1}{3}$, then Player 1 prefers Back $(p = 0)$; otherwise Center.
- **Player 2:** If $p > \frac{1}{4}$, then Player 2 prefers Center $(q = 1)$; otherwise Back.

Tennis game (msNE with three available strategies)

- Best Responses in the Tennis Game
    - **Player 1**

$$BR_1(q) = \begin{cases} \text{Back if } q > \frac{1}{4} \\ \{\text{Center, Back}\} \text{ if } q = \frac{1}{4} \\ \text{Center if } q < \frac{1}{4} \end{cases}$$

        - (Recall that $p = 0$ implies playing strategy back with probability one).
    - **Player 2**

$$BR_2(p) = \begin{cases} \text{Center if } p > \frac{1}{4} \\ \{\text{Center, Back}\} \text{ if } p = \frac{1}{4} \\ \text{Back if } p < \frac{1}{4} \end{cases}$$

Graphical representation of BRFs and msNE:

1. Matching pennies (Done ✓)
2. Battle of the sexes (coordination) (Done ✓)
3. Additional practice:
    1. Lobbying game (Watson page 124).
    2. Chicken game (anticoordination).

A few tricks we just learned...

- **Indifference:** If it is optimal to randomize over a collection of pure strategies, then a player receives the same expected payoff from each of those pure strategies.
    - He must be indifferent between those pure strategies over which he randomizes.
- **Odd number:** In almost all finite games (games with a finite set of players and available actions), there is a finite and odd number of equilibria.
    - *Examples*: 1 NE in matching pennies (only one msNE), 3 NE in BoS (two psNE, one msNE), 1 in PD (only one psNE), etc.
- **Never use strictly dominated strategies:** If a pure strategy does not survive the IDSDS, then a NE assigns a zero probability to that pure strategy.
    - *Example*: PD game, where NC is strictly dominated, it does not receive any positive probability.

What if players have three undominated strategies?

- Consider the rock-paper-scissors game

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0, 0	-1, $\underline{1}$	$\underline{1}$, -1
	Paper	$\underline{1}$, -1	0, 0	-1, $\underline{1}$
	Scissors	-1, $\underline{1}$	$\underline{1}$, -1	0, 0

- First, note that neither player selects a pure strategy (with 100% probability).

What if players have three undominated strategies?

- Second, every player must be mixing between all his three possible actions, R, P and S.

If Player 1 only mixes between Rock and Paper

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

- **Otherwise:** if P1 mixes only between Rock and Paper, then Player 2 prefers to respond with Paper rather than Rock.
- But if Player 2 never uses Rock, then Player 1 gets a higher payoff with Scissors than Paper. **Contradicton!**
- Then players cannot be mixing between only two of their available strategies.

What if players have three undominated strategies?

- Are you suspecting that the msNE is $\sigma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$? You're right!

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

- We must make every player indifferent between using Rock, Paper, or Scissors.
- That is, $u_1(Rock, \sigma_2) = u_1(Paper, \sigma_2) = u_1(Scissors, \sigma_2)$ for Player 1, and
- $u_2(\sigma_1, Rock) = u_2(\sigma_1, Paper) = u_2(\sigma_1, Scissors)$ for Player 2.

What if players have three undominated strategies?

- Let's separately find each of these expected utilities.
- If player 1 chooses Rock (first row), he obtains

$$\begin{aligned} u_1(Rock, \sigma_2) &= 0\sigma_2(R) + (-1)\sigma_2(P) + 1(1 - \sigma_2(R) - \sigma_2(P)) \\ &= -1\sigma_2(P) + 1 - \sigma_2(R) - \sigma_2(P) \end{aligned}$$

First Row

		Player 2		
		$\sigma_2(R)$ Rock	$\sigma_2(P)$ Paper	$1 - \sigma_2(R) - \sigma_2(P)$ Scissors
	Rock	0, 0	-1, 1	1, -1
Player 1	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

What if players have three undominated strategies?

- If player 1 chooses Paper (second row), he obtains

$$\begin{aligned} u_1(Paper, \sigma_2) &= 1\sigma_2(R) + 0\sigma_2(P) + (-1)(1 - \sigma_2(R) - \sigma_2(P)) \\ &= \sigma_2(R) - 1 + \sigma_2(R) + \sigma_2(P) \end{aligned}$$

Player 2

Second Row

		$\sigma_2(R)$ Rock	$\sigma_2(P)$ Paper	$1 - \sigma_2(R) - \sigma_2(P)$ Scissors
	Rock	0, 0	-1, 1	1, -1
Player 1	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

What if players have three undominated strategies?

- If player 1 chooses Scissors (third row), he obtains

$$\begin{aligned} u_1(Scissors, \sigma_2) &= (-1)\sigma_2(R) + 1\sigma_2(P) + 0(1 - \sigma_2(R) - \sigma_2(P)) \\ &= -\sigma_2(R) + \sigma_2(P) \end{aligned}$$

		Player 2		
		$\sigma_2(R)$ Rock	$\sigma_2(P)$ Paper	*1* - $\sigma_2(R)$ - $\sigma_2(P)$ Scissors
	Rock	0, 0	-1, 1	1, -1
Player 1	Paper	1, -1	0, 0	-1, 1
Third Row	Scissors	-1, 1	1, -1	0, 0

What if players have three undominated strategies?

- Making the three expected utilities

$$\begin{aligned} u_1(Rock, \sigma_2) &= -1\sigma_2(P) + 1 - \sigma_2(R) - \sigma_2(P), \\ u_1(Paper, \sigma_2) &= \sigma_2(R) - 1 + \sigma_2(R) + \sigma_2(P), \text{ and} \\ u_1(Scissors, \sigma_2) &= -\sigma_2(R) + \sigma_2(P) \end{aligned}$$

  equal to each other, we obtain

$$\sigma_2(R) = \sigma_2(P) = 1 - \sigma_2(R) - \sigma_2(P)$$

- Hence, player 2 assigns the same probability weights to his three available actions, thus implying

$$\sigma_2^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

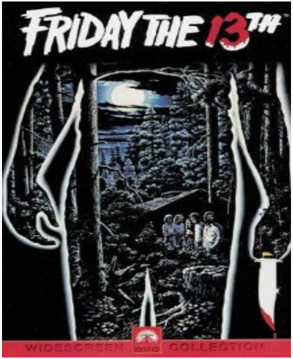
- A similar argument is applicable to player 1, since players' payoffs are symmetric.

Summarizing...

- We learned how to find msNE in games:
- with 2 players, each with 2 available strategies (2x2 matrix)
    - e.g., matching pennies game, battle of the sexes, etc.
- with 2 players, but each having 3 available strategies (3x3 matrix)
    - e.g., tennis game (which actually reduced to a 2x2 matrix after deleting strictly dominated strategies), and
    - the rock-paper-scissors game, where we couldn't identify strictly dominated strategies and, hence, had to make players indifferent between their three available strategies.
- What about games with 3 players?

More advanced mixed strategy games

What if we have three players, instead of two? (Harrington pp 201-204). "Friday the 13th!"



More advanced mixed strategy games

Beth

		Front	Back
Tommy	Front	0, 0, $\underline{0}$	-4, $\underline{1}$, $\underline{2}$
	Back	$\underline{1}$, -4, $\underline{2}$	$\underline{2}$, $\underline{2}$, -2

Jason, Front

Beth

		Front	Back
Tommy	Front	$\underline{3}$, $\underline{3}$, -2	$\underline{1}$, -4, $\underline{2}$
	Back	-4, $\underline{1}$, $\underline{2}$	0, 0, $\underline{0}$

Jason, Back

More advanced mixed strategy games

Friday the 13th!

		Beth	
		Front	Back
Tommy	Front	0, 0, $\underline{0}$	-4, $\underline{1}$, $\underline{2}$
	Back	$\underline{1}$, -4, $\underline{2}$	$\underline{2}$, $\underline{2}$, -2

Jason, Front

		Beth	
		Front	Back
Tommy	Front	$\underline{3}$, $\underline{3}$, -2	$\underline{1}$, -4, $\underline{2}$
	Back	-4, $\underline{1}$, $\underline{2}$	0, 0, $\underline{0}$

Jason, Back

1. **First step:** let's check for strictly dominated strategies (none).
2. **Second step:** let's check for psNE (none). The movie is getting interestin!
3. **Third step:** let's check for msNE. (note that all strategies are used by all players), since there are no strictly dominated strategies.

msNE with three players

- Since we could not delete any strictly dominated strategy, then all strategies must be used by all three players.
- In this exercise we need three probabilities, one for each player.
- Let's denote:
    - t the probability that Tommy goes through the front door (first row in both matrices).
    - b the probability that Beth goes through the front door (first column in both matrices).
    - j the probability that Jason goes through the front door (left-hand matrix).

msNE with three players

Let us start with **Jason**, $EU_J(F) = EU_J(B)$, where

$$\begin{aligned} EU_J(F) &= \underbrace{tb0 + t(1-b)2}_{\substack{\text{Tommy goes through} \\ \text{the front door, } t}} + \underbrace{(1-t)b2 + (1-t)(1-b)(-2)}_{\substack{\text{Tommy goes through} \\ \text{the back door, } (1-t)}} \\ &= -2 + 4t + 4b - 6tb \end{aligned}$$

and

$$\begin{aligned} EU_J(B) &= tb(-2) + t(1-b)2 + (1-t)b2 + (1-t)(1-b)0 \\ &= 2t + 2b - 6tb \end{aligned}$$

since $EU_J(F) = EU_J(B)$ we have

$$-2 + 4t + 4b - 6tb = 2t + 2b - 6tb \iff \underbrace{t + b = 1}_{\text{Condition (1)}} \qquad (1)$$

msNE with three players

Let us now continue with **Tommy**, $EU_T(F) = EU_T(B)$, where

$$\begin{aligned} EU_T(F) &= bj0 + (1-b)j(-4) + b(1-j)3 + (1-b)(1-j)(1) \\ &= 1 + 2b - 5j + 2bj \end{aligned}$$

and

$$\begin{aligned} EU_T(B) &= bj1 + (1-b)j2 + b(1-j)(-4) + (1-b)(1-j)(0) \\ &= -4b + 2j + 3bj \end{aligned}$$

since $EU_T(F) = EU_T(B)$ we have

$$1 + 2b - 5j + 2bj = -4b + 2j + 3bj \iff \underbrace{7j - 6b + bj = 1}_{\text{Condition (2)}} \quad (2)$$

msNE with three players

- And given that the payoffs for Tommy and Beth are symmetric, we must have that Tommy and Beth's probabilities coincide, $t = b$.
    - Hence we don't need to find the indifference condition $EU_B(F) = EU_B(B)$ for Beth.
    - Instead, we can use Tommy's condition (2) (i.e., $7j - 6b + bj = 1$), to obtain the following condition for Beth:

$$7j - 6t + tj = 1$$

- We must solve conditions (1),(2) and (3).

- First, by symmetry we must have that $t = b$. Using this result in condition (1) we obtain

$$t + b = 1 \implies t + t = 1 \implies t = b = \frac{1}{2}$$

- Using this result into condition (2), we find

$$7j - 6b + bj = 7j - 6\frac{1}{2} + \frac{1}{2}j = 1$$

Solving for j we obtain $j = \frac{8}{15}$.

msNE with three players

- Representing the msNE in Friday the 13th:

$$\left\{ \underbrace{\left(\frac{1}{2}\text{Front},\ \frac{1}{2}\text{Back}\right)}_{\text{Tommy}}, \underbrace{\left(\frac{1}{2}\text{Front},\ \frac{1}{2}\text{Back}\right)}_{\text{Beth}}, \underbrace{\left(\frac{8}{15}\text{Front},\ \frac{7}{15}\text{Back}\right)}_{\text{Jason}} \right\}$$

msNE with three players

- **Just for fun:** What is then the probability that Tommy and Beth scape from Jason?
  - They scape if they both go through a door where Jason is not located.

$$\frac{1}{2}\frac{1}{2}\underbrace{\frac{8}{15}}_{\text{Jason goes Front}}+\frac{1}{2}\frac{1}{2}\underbrace{\frac{7}{15}}_{\text{Jason goes Back}}=\frac{15}{60}$$

- The **first term** represents the probability that both Tommy and Beth go through the Back door (which occurs with $\frac{1}{2}\frac{1}{2}=\frac{1}{4}$ probability) while Jason goes to the Front door.
- The **second term** represents the opposite case: Tommy and Beth go through the Front door (which occurs with $\frac{1}{2}\frac{1}{2}=\frac{1}{4}$ probability) while Jason goes to the Back door.

msNE with three players

- Even if they escape from Jason this time, there is still...

- There are actually NO sequels:
  - Their probability of escaping Jason is then $(\frac{15}{60})^{10}$,about 1 in a million !

Testing the Theory

- A natural question at this point is how we can empirically test, as external observers, if individuals behave as predicted by our theoretical models.
    - In other words, how can we check if individuals randomize with approximately the same probability that we found to be optimal in the msNE of the game?

Testing the Theory

- In order to test the theoretical predictions of our models, we need to find settings where players seek to "surprise" their opponents (so playing a pure strategy is not rational), and where stakes are high.
  - Can you think of any?

Penalty kicks in soccer

Penalty kicks in soccer

His payoffs represent the probability that the kicker does not score (That is why within a given cell, payoffs sum up to one).

Payoffs represent the probability he scores.

Goalkeeper

		Left	Center	Right
	Left	.65, .35	.95, .05	.95, .05
Kicker	Center	.95, .05	0, 1	.95, .05
	Right	.95, .05	.95, .05	.65, .35

Penalty kicks in soccer

- We should expect soccer players randomize their decision.
  - Otherwise, the kicker could anticipate where the goalie dives and kick to the other side. Similarly for the goalie.
- Let's describe the kicker's expected utility from kicking the ball left, center or right.

Penalty kicks in soccer

$$\begin{aligned} EU_{\text{Kicker}}(Left) &= g_l * 0.65 + g_r * 0.95 + (1 - g_r - g_l) * 0.95 \\ &= 0.95 - 0.3 g_l \end{aligned} \tag{1}$$

$$\begin{aligned} EU_{\text{Kicker}}(Center) &= g_l * 0.95 + g_r * 0.95 + (1 - g_r - g_l) * 0 \\ &= 0.95(g_r + g_l) \end{aligned} \tag{2}$$

$$\begin{aligned} EU_{\text{Kicker}}(Right) &= g_l * 0.95 + g_r * 0.65 + (1 - g_r - g_l) * 0.95 \\ &= 0.95 - 0.3 g_r \end{aligned} \tag{3}$$

Penalty kicks in soccer

- Since the kicker must be indifferent between all his strategies, $EU_{\text{Kicker}}(Left) = EU_{\text{Kicker}}(Right)$

  $$0.95 - 0.3g_l = 0.95 - 0.3g_r \implies g_l = g_r \implies g_l = g_r = g$$

  Using this information in (2), we have

  $$0.95(g+g) = 1.9g$$

  Hence,

  $$\underbrace{0.95 - 0.3g}_{\substack{EU_{\text{Kicker}}(Left) \\ \text{or} \\ EU_{\text{Kicker}}(Right)}} = \underbrace{1.9g}_{EU_{\text{Kicker}}(Center)} \implies g = \frac{0.95}{2.2} = 0.43$$

Penalty kicks in soccer

- Therefore,

$$(\sigma_L, \sigma_C, \sigma_R) = (\underbrace{0.43}_{g_l}, \underbrace{0.14}_{\substack{\text{From the fact that} \\ g_l+g_r+g_c=1}}, \underbrace{0.43}_{\substack{g_r, \\ \text{where } g_l=g_r=g}})$$

- If the set of goalkeepers is similar, we can find the same set of mixed strategies,

$$(\sigma_L, \sigma_C, \sigma_R) = (0.43, 0.14, 0.43)$$

Penalty kicks in soccer

- Hence, the probability that a goal is scored is:
    - Goalkeeper dives left ⟶

$$0.43 * (\underbrace{0.43}_{\substack{\text{Kicker}\\\text{aims}\\\text{left}}} *0.65 + \underbrace{0.14}_{\substack{\text{Kicker}\\\text{aims}\\\text{center}}} *0.95 + \underbrace{0.43}_{\substack{\text{Kicker}\\\text{aims}\\\text{right}}} *0.95)$$

    - Goalkeeper dives center ⟶

$$+0.14 * (0.43 * 0.95 + 0.14 * 0 + 0.43 * 0.95)$$

    - Goalkeeper dives right ⟶

$$+0.43 * (0.43 * 0.95 + 0.14 * 0.95 + 0.43 * 0.65)$$

$= 0.82044$, i.e., a goal is scored with 82% probability.

Penalty kicks in soccer

- Interested in more details?
  - First, read Harrington pp. 199-201.
  - Then you can have a look at the article
    - "Professionals play Minimax" by Ignacio Palacios-Huerta, *Review of Economic Studies*, 2003.
  - This author published a very readable book last year:
    - *Beautiful Game Theory: How Soccer Can Help Economics.* Princeton University Press, 2014.

Summarizing...

- So far we have learned how to find msNE is games:
    - with two players (either with 2 or more available strategies).
    - with three players (e.g., Friday the 13th movie).
- What about generalizing the notion of msNE to games with N players?
    - Easy! We just need to guarantee that every player is indifferent between all his available strategies.

msNE with N players

- **Example: "Extreme snob effect" (Watson).**
- Every player chooses between alternative X and Y (Levi's and Calvin Klein). Every player i's payoff is 1 if he selects Y, but if he selects X his payoff is:
    - 2 if no other player chooses X, and
    - 0 if some other player chooses X as well

- Let's check for a symmetric msNE where all players select Y with probability α. Given that player i must be indifferent between X and Y, $EU_i(X) = EU_i(Y)$, where

$$EU_i(X) = \underbrace{\alpha^{n-1}2}_{\text{all other } n-1 \text{ players select } Y} + \underbrace{(1-\alpha^{n-1})0}_{\text{Not all other players select Y}}$$

msNE with N players

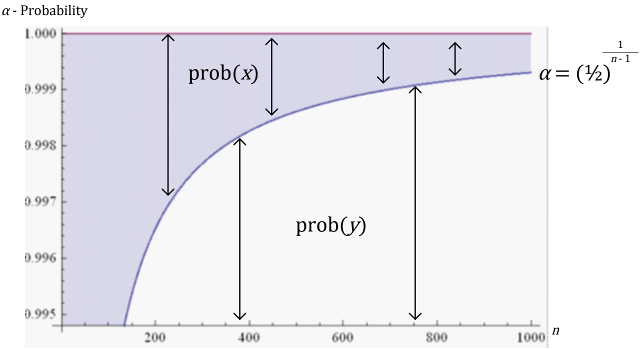
- and $EU_i(Y) = 1$, then $EU_i(X) = EU_i(Y)$ implies

$$\alpha^{n-1}2 = 1 \iff \alpha = \left(\frac{1}{2}\right)^{\frac{1}{n-1}}$$

- **Comparative statics of α, the probability a player selects the "conforming" option Y, $\alpha = \left(\frac{1}{2}\right)^{\frac{1}{n-1}}$:**
    - α increases in the size of the population n.
        - That is, the larger the size of the population, the more likely it is that somebody else chooses the same as you, and as a consequence you don't take the risk of choosing the snob option X. Instead, you select the "conforming" option Y.

msNE with N players

- Probability of choosing strategy Y as a function of the number of individuals, *n*.



$$prob(X) + prob(Y) = 1, prob(X) ...then, (X) = 1 - prob(Y)$$

Another example of msNE with N players

- **Another example with N players: The bystander effect**
- The "bystander effect" refers to the lack of response to help someone nearby who is in need.
  - *Famous example*: In 1964 Kitty Genovese was attacked near her apartment building in New York City. Despite 38 people reported having heard her screams, no one came to her aid.
  - Also confirmed in laboratory and field studies in psychology.

Another example of msNE with N players

- General finding of these studies:
  - A person is less likely to offer assistance to someone in need when the person is in a large group than when he/she is alone.
    - e.g., all those people who heard Kitty Genovese's cries knew that many others heard them as well.
  - In fact, some studies show that the *more* people that are there who could help, the *less* likely help is to occur.
- Can this outcome be consistent with players maximizing their utility level?
  - Yes, let's see how.

Another example of msNE with N players

		Other players	
		All ignore	At least one helps
Player	Helps	$\underline{a}$	c
	Ignores	d	$\underline{b}$

- where $a > d \longrightarrow$ so if all ignore, I prefer to help the person in need.
- but $b > c \longrightarrow$ so, if at least somebody helps, I prefer to ignore.
- Note that assumptions are not so selfish : people would prefer to help if nobody else does.

Another example of msNE with N players

- msNE:
  - Let's consider a *symmetric msNE* whereby every player i:
    - Helps with probability p, and
    - Ignores with probability $1 - p$.

Another example of msNE with N players

$$
\begin{aligned}
EU_i(Help) &= \underbrace{(1-p)^{n-1} * a}_{\substack{\text{If everybody}\\ \text{else ignores}}} + \underbrace{\left[1-(1-p)^{n-1}\right] * c}_{\substack{\text{If at least one of the}\\ \text{other } n-1 \text{ players helps}}} \\
EU_i(Ignore) &= \underbrace{(1-p)^{n-1} * d}_{\substack{\text{If everybody}\\ \text{else ignores}}} + \underbrace{\left[1-(1-p)^{n-1}\right] * b}_{\substack{\text{If at least one of the}\\ \text{other } n-1 \text{ players helps}}}
\end{aligned}
$$

- When a player randomizes, he is indifferent between help and ignore,

$$
\begin{aligned}
EU_i(Help) &= EU_i(Ignore) \\
& \quad (1-p)^{n-1} * a + \left[1-(1-p)^{n-1}\right] * c \\
&= (1-p)^{n-1} * d + \left[1-(1-p)^{n-1}\right] * b \\
&\implies (1-p)^{n-1}(a-c-d+b) = b-c
\end{aligned}
$$

Another example of msNE with N players

- Solving for p,

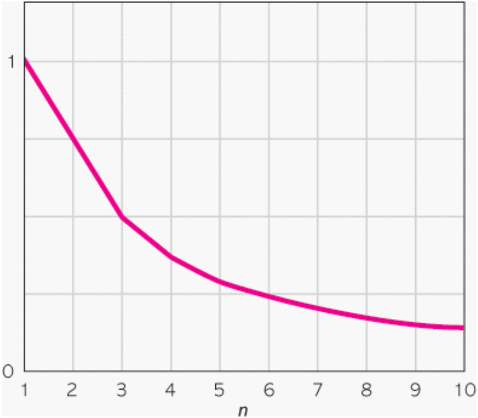
$$\begin{aligned} (1-p)^{n-1} &= \frac{b-c}{a-c-d+b} \\ &\implies 1-p = \left(\frac{b-c}{a-c-d+b}\right)^{\frac{1}{n-1}} \\ &\implies p^* = 1-\left(\frac{b-c}{a-c-d+b}\right)^{\frac{1}{n-1}} \end{aligned}$$

- Example: $a = 4$, $b = 3$, $c = 2$, $d = 1$, satisfying the initial assumptions: $a > d$ and $b > c$

$$p^* = 1-\left(\frac{3-1}{4-2-1+3}\right)^{\frac{1}{n-1}} = 1-\left(\frac{1}{4}\right)^{\frac{1}{n-1}}$$

Another example of msNE with N players

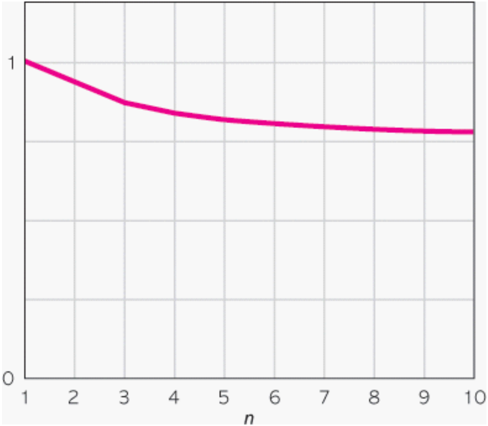
- Probability of a person helping, p^*



More people makes me less likely to help.

Another example of msNE with N players

- Probability that the person in need receives help, $(p^*)^n$



More people actually make it **less** likely that the victim is helped!

- Intuitively, the new individual in the population brings a positive and a negative effect on the probability that the victim is finally helped:
    - **Positive effect**: the additional individual, with his own probability of help, p^*, increases the chance that the victim is helped.
    - **Negative effect**: the additional individual makes more likely, that someone will help the victim,thus leading each individual citizen to reduce his own probability of helping, i.e., p^* decreades in n.
- However, the fact that $(p^*)^n$ decreases in n implies that the negative effect offsets the positive effect.