Applications of Game Theory to Environmental Problems

Ana Espinola-Arredondo

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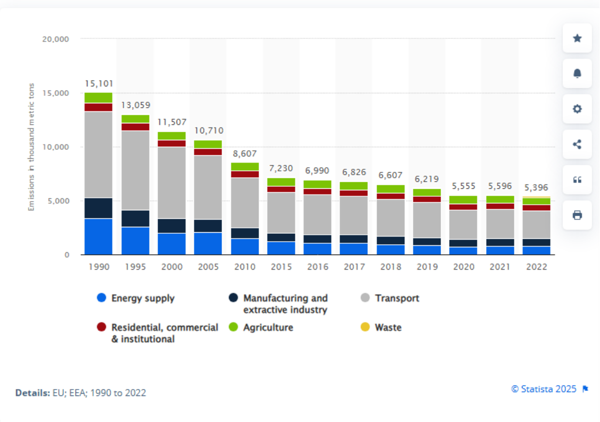
Introduction

- Exam Form A - Portfolio:
    - Assignment 1: Game Theory Concepts (week 1).
    - Assignment 2: Case Study I (CPR and Market Based Policy, week 2)
    - Assignment 3: Case Study II (Abatement and Labelling, week 3)
- Game Theory: An Introduction with Step By-Step Examples, A. Espinola-Arredondo and F. Munoz-Garcia, Palgrave MacMillan, December 2023.
- Games, Strategies and Decision Making. Joseph Harrington Jr. Worth Publishers. (Second edition) 2014.

Introduction

- Total nitrogen oxide (NOx) emissions in the European Union (EU-27) from 1990 to 2022, by sector (in 1,000 metric tons)



Details: EU; EEA; 1990 to 2022

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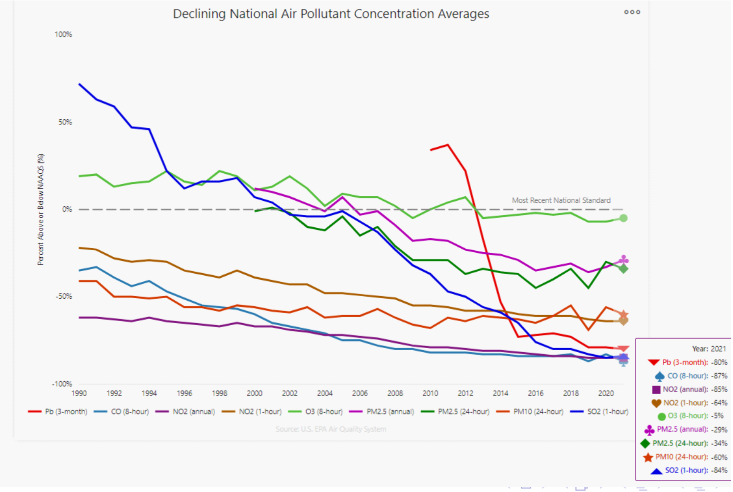
Introduction

Our Nation's Air 2022 (epa.gov)



Source: U.S. EPA Air Quality System

Introduction

Measuring the damages of air pollution in the United States (Muller and Mendelsohn, 2007, JEEM)

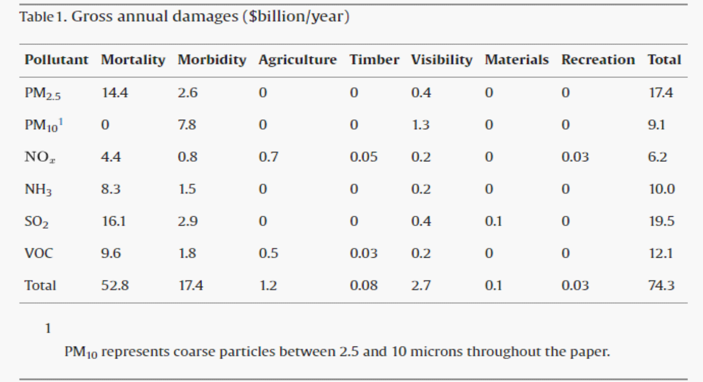


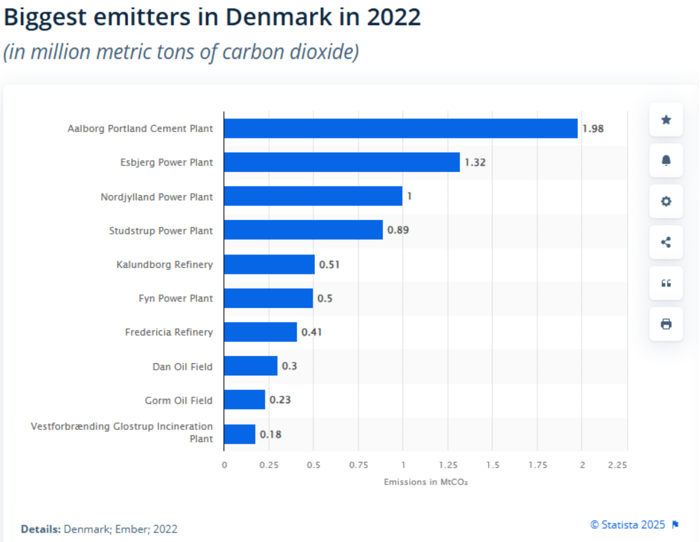
Table 1. Gross annual damages ($billion/year)

Pollutant	Mortality	Morbidity	Agriculture	Timber	Visibility	Materials	Recreation	Total
$PM_{2.5}$	14.4	2.6	0	0	0.4	0	0	17.4
PM_{10}[1]	0	7.8	0	0	1.3	0	0	9.1
NO_x	4.4	0.8	0.7	0.05	0.2	0	0.03	6.2
NH_3	8.3	1.5	0	0	0.2	0	0	10.0
SO_2	16.1	2.9	0	0	0.4	0.1	0	19.5
VOC	9.6	1.8	0.5	0.03	0.2	0	0	12.1
Total	52.8	17.4	1.2	0.08	2.7	0.1	0.03	74.3

[1] PM_{10} represents coarse particles between 2.5 and 10 microns throughout the paper.

- The largest source of SO2 in the atmosphere is the burning of fossil fuels by power plants and other industrial facilities.

Introduction



Biggest emitters in Denmark in 2022

(in million metric tons of carbon dioxide)

Details: Denmark; Ember; 2022

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Introduction

- Air pollution causes decreases in happiness and increases in depression.
- Research has shown that people living in places with excessive amounts of PM2.5 have a heightened risk for dementia by 92%.
- Cognitively, it impairs functioning and decision-making.
- Economically, it hurts work productivity. And socially, it exacerbates criminal behavior.
- In a study that analyzed a nine-year panel of 9,360 U.S. cities, air pollution positively predicted both violent crimes (murder, rape, robbery, and assault) and property crimes (burglary and motor vehicle theft).

Introduction

- If the market won't solve its own externality problems, then what?
- The three types of policies governments use to deal with pollution:

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- tradable emissions permits.

- Game Theory as the study of interdependence
    - "No man is an island"
- Definition:
    - Game Theory: a formal way to analyze **interaction** among a **group** of **rational** agents who behave strategically.

- Several important elemants of this definition help us understand what is game theory, and what is not:
- **Interaction:** If your actions do not affect anybody else, that is not a situation of interdependence.
- **Group:** we are not interested in games you play with your imaginary friend, but with other people, firms, etc.
- **Rational agents:** we assume that agents will behave rationally especially if the stakes are high and you allow them sufficient time to think about their available strategies.
  - Although we mention some experiments in which individuals do not behave in a completely rational manner...
  - these "anomalies" tend to vanish as long as you allow for sufficient repetitions, i.e., everybody ends up learning, or you raise stakes sufficiently (high incentives).

Examples (1):

- Output decision of two competing firms:
    - Cournot model of output competition.
- Research and Development expenditures:
    - They serve as a way to improve a firm's competitiveness in posterior periods.
- OPEC pricing, how to sustain collusion in the long run...

Examples (2):

- Sustainable use of natural resources *and* overexploitation of the common resource.
- Use of environmental policy as a policy to promote exports.
    - Setting tax emission fees in order to favor domestic firms.
- Public goods (everybody wants to be a "free-rider").
    - I have never played a public good game!
    - Are you sure? A group project in class. The slacker you surely faced was our "free-rider."

Rules of a General Game (informal):(WATSON CH.2,3)

The rules of a game seek to answer the following questions:

1. Who is playing ?$\leftarrow$ set of players (I)
2. What are they playing with ?$\leftarrow$ Set of available actions (S)
3. Where each player gets to play ?$\leftarrow$ Order, or time structure of the game.
4. How much players can gain (or lose) ?$\leftarrow$ Payoffs (measured by a utility function $U_i(s_i, s_{-i})$

1. We assume **Common knowledge** about the rules of the game.

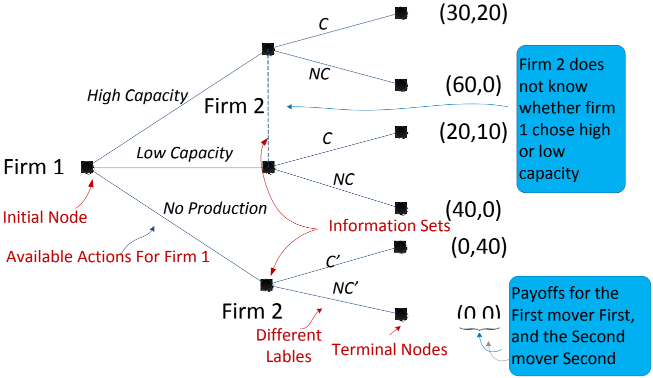
- As a player, I know the answer to the above four questions (rules of the game)
- In addition, I know that you know the rules, and...
- that you know that I know that you know the rules,.....(ad infinitum).

Two ways to graphically represent games

- Extensive form
    - We will use a game tree (next slide).
- Normal form (also referred as "strategic form").
    - We will use a matrix.

Example of a game tree

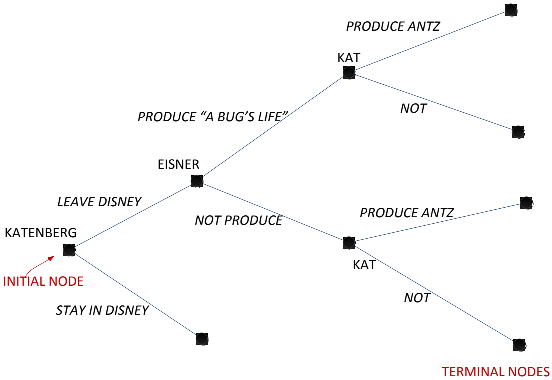
- Consider the following sequential-move game played by firms 1 and 2:
  - We will use a matrix



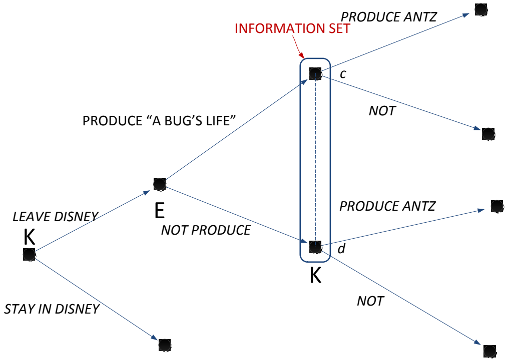
Firm 2 does not know whether firm 1 chose high or low capacity

Payoffs for the First mover First, and the Second mover Second

"ANTZ" vs. "A BUG'S LIFE"



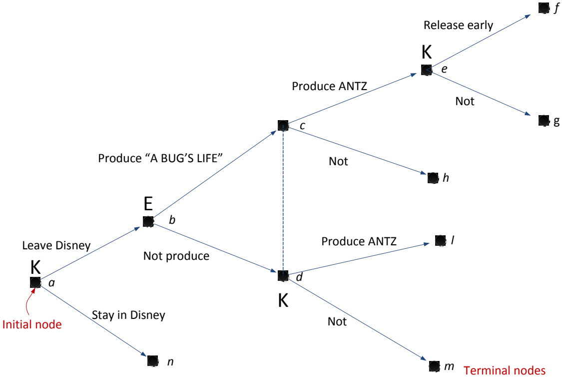
- In this example, Katsenberg observes whether Eisner produced the film "A BUG"S LIFE" or not before choosing to produce "ANTZ".



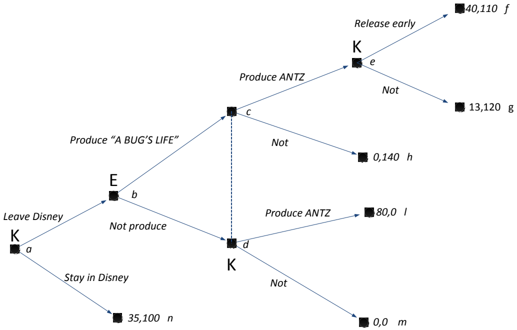
- When Katsenberg is at the move (either at node c or d), he knows that he is at one of these nodes,but he does not know at which one *and* the figure captures this lack of information with a dashed line connecting the nodes.:

The Bug Game

- We now add an additional stage at the end at which Katsenberg is allowed to release "Antz" early in case he produced the movie and Eisner also produced "A bug's life" (at node e).



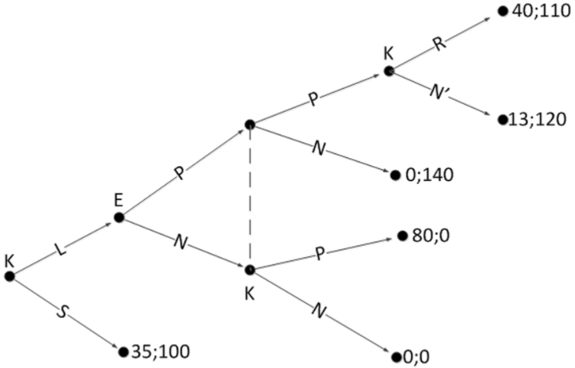
The Extensive Form of The Bug Game



- Let's define the payoff numbers as the profits that each obtains in the various outcomes, i.e.,in each terminal node.
- For example, in the event that Katzenberg stays at Disney, we assume he gets $35 million and Eisner gets $100 million (terminal node a).

The Bug Game Extensive Form (Abbreviating Labels)

- We often abbreviate labels in order to make the figure of the game tree less jammed, as we do next.



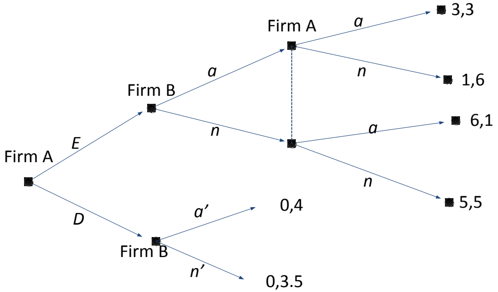
Information sets

- An information set is graphically represented with two or more nodes connected by a dashed line, (or a "sausage") including all these connected nodes.
- It represents that the player called to move at that information set cannot distinguish between the two or more actions chosen by his opponent before he is called to move.
- Hence, the set of available actions must be the same in all the nodes included on that information set (P and N in the previous game tree for Katsenberg).
    - Otherwise, Katsenberg, despite not observing Eisner's choice, would be able to infer it by analyzing which are the available actions he can choose from.

Guided exercise (page 19-20 in Watson)

- **Lets practice how to depict a game tree of a strategic situation on an industry:**
- Firm A decides whether to enter firm B's industry. Firm B observes this decision.
  - If firm A stays out, firm B alone decides whether to advertise. In this case, firm A obtains zero profits, and firm B obtains $4 million if it advertises and $3.5 million if it does not.
  - If firm A enters, both firms simultaneously decide whether to advertise, obtaining the following payoffs.
    - If both advertise, both firms earn $3 million.
    - If none of them advertise, both firms earn $5 million.
    - If only one firm advertises, then it earns $6 million and the other firm earns $1 million.

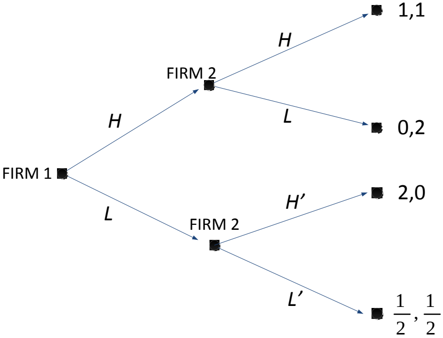
Guided Exercise, (continued)



- Let E and D denote firm A's initial alternatives of entering and not entering B's industry.
- Let a and n stand for "advertise" and "not advertise", respectively.
- Note that simultaneous advertising decisions are captured by firm A's information set.

Strategy: Definition of Strategy

- Lets practice finding the strategies of firm 1 and 2 in the following game tree:
    - We will use a matrix



Strategies for firm 1 : H and L.
Strategies for firm 2 : H. H’;H. L’;L. H;L

Strategy space and Strategy profile

- **Strategy space:** It is a set comprising each of the possible strategies of player i.
  - From our previous example:
    - $S_1 = \{H, L\}$ for firm 1
    - $S_2 = \{HH', HL', LH', LL'\}$ for firm 2.
- **Strategy profile**
  - It is a vector (or list) describing a particular strategy for every player in the game. For instance, in a two-player game

    $$s = (s_1, s_2)$$

    where s_1 is a specific strategy for firm 1.(for instance, $s_1 = H$), and s_2 is a specific strategy for firm 2, e.g., $s_2 = LH'$.
  - More generally, for N players, a strategy profile is a vector with N components,

    $$s = (s_1, s_2, s_3, ..., s_n)$$

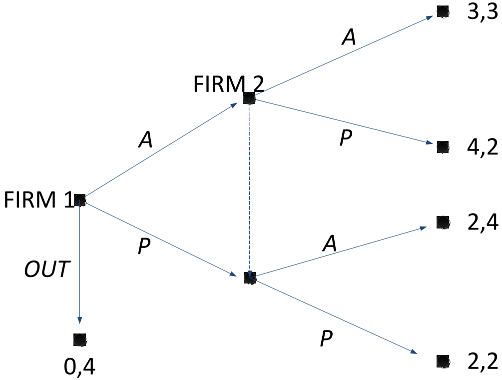
Strategy profile:

- In order to represent the strategies selected by all players except player i, we write:

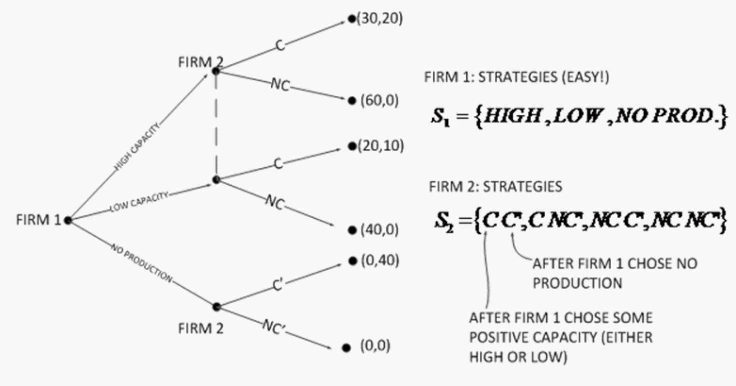
  $$s_{-i} = (s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n)$$

  (Note that these strategies are potentially different)
- We can hence write, more compactly, as strategy profile with only two elements:
  The strategy player i selects, s_i, and the strategies chosen by everyone else, s_{-i}, as : $s = (s_i, s_{-i})$
- **Example:**
  - Consider a strategy profile s which states that player 1 selects B, player 2 chooses X, and player 3 selects Y, i.e., $s = (B, X, Y)$.Then,
    - $s_{-1} = (X, Y)$,
    - $s_{-2} = (B, Y)$, and
    - $s_{-3} = (B, X)$.

- Lets practice finding strategy sets in the following game tree:



- Let's define firm 1 and 2's available strategies in the first example of a game tree we described a few minutes ago:



FIRM 1: STRATEGIES (EASY!)

$S_1 = \{HIGH, LOW, NO\ PROD.\}$

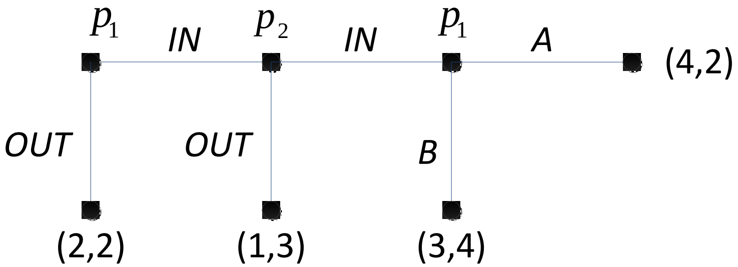
FIRM 2: STRATEGIES

$S_2 = \{CC', CNC', NCC', NCNC'\}$

AFTER FIRM 1 CHOSE NO PRODUCTION

AFTER FIRM 1 CHOSE SOME POSITIVE CAPACITY (EITHER HIGH OR LOW)

ANOTHER EXAMPLE: THE CENTIPEDE GAME:



- Strategy set for player 2 : S_2 = {IN, OUT}
- Strategy set for player 1 : S_1 = {IN A, IN B, OUT A, OUT B}
- More examples on page 27 (Watson)

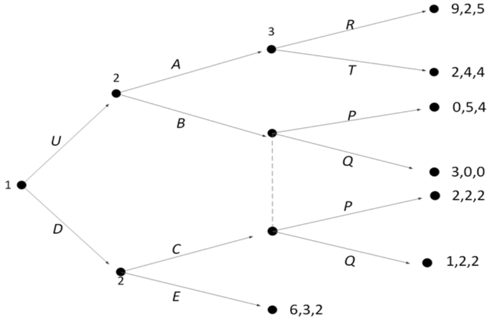
One second...

- Why do we have to specify my future actions after selecting "out" ? Two reasons:

1. Because of potential mistakes:
    - Imagine I ask you to act on my behalf, but I just inform you to select "out" at the initial node. However, you make a mistake (i.e., you play "In"), and player 2 responds with "In" as well. What would you do now??
    - With a strategy (complete contingent plan) you would know what to do even in events that are considered off the equillibrium path.
2. Because player 1's action later on affects player 2's actions, and ...
    - ultimately player 2's actions affects player 1's decision on whether to play "In" or "Out" at the beginning of the game.
    - This is related with the concept of backwards induction that we will discuss when solving sequential-move games.)

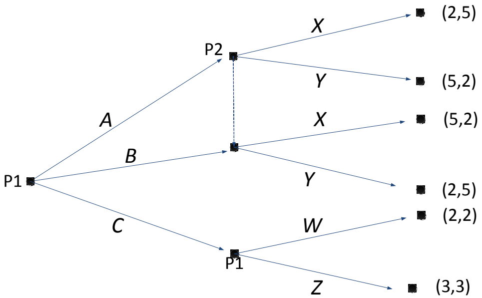
Some extensive-form games

- Let's now find the strategy spaces of a game with three players:



- S_1 = {U, D}
- S_2 = {AC, AE, BC, BE}; and
- S_3 ={RP,RQ,TP,TQ}

Some extensive-form games (Cont'l)



- $S_1 = \{AW, BW, CW, AZ, BZ, CZ\}$
- $S_2 = \{X, Y\}$

- **When a game is played simultaneously, we can represent it using a matrix**
    - *Example:* Prisoners' Dilemma game.

		Prisoner 2	
		Confess	Don't Confess
Prisoner 1	Confess	$-5, -5$	$0, -15$
	Don't Confess	$-15, 0$	$-1, -1$

- **Another example of a simultaneous-move game**
  - The "battle of the sexes" game.(I know the game is sexist, but please don’t call it the "battle of the sexist" game !)

		Wife	
		Opera	Movie
Husband	Opera	1, 2	0, 0
	Movie	0, 0	2, 1

- **Yet, another example of a simultaneous-move game**
  - Pareto-coordination game.

		Firm 2	
		Superior tech.	Inferior tech.
Firm 1	Superior tech.	2, 2	0, 0
	Inferior tech.	0, 0	1, 1

- **Yet, another example of a simultaneous-move game**
    - The game of "chicken."

		Dean	
		Straight	Swerve
James	Straight	0, 0	3, 1
	Swerve	1, 3	2, 2

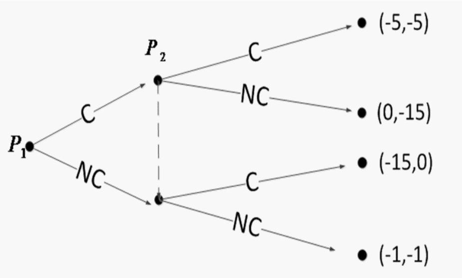
Other examples of the "Chickengame"

Mode	Description
Trackors	Footloose, (1984,Movie)
Bulldozers	Buster and Gob in Arrested Development (2004,TV)
Wheelchairs	Two old ladies with motorized wheelchairs in Banzai(2003,TV)
Snowmobiles	"[Two adult males] died in a head-on collision, earning a tie in the game of chicken they were playing with their snowmobiles" <www.seriouslyinternet.com/278.0.html>
Film Release Dates	Dreamworks and Disney-PIxar (2004)
Nuclear Weapons	Cuban Missile Crisis (1963)

Normal (Strategic) Form

- We can alternatively represent simultaneous-move games using a game tree, as long as we illustrate that players choose their actions without observing each others' moves, i.e., using information sets, as we do next for the prisoner's dilemma game:
- Extensive form representation of the Prisoner's Dilemma game :

		P_2	
		C	NC
P_1	C	-5,-5	0,-15
	NC	-15,0	-1,-1

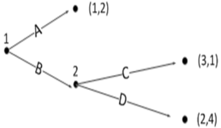


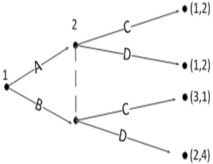
- **Practice** :Using a game tree, depict the equivalent extensive - form representation of the following matrix representing the "Battle of the Sexes" game.

		Wife	
		Opera	Movie
Husband	Opera	1,2	0,0
	Movie	0,0	2,1

Corresponding extensive and normal forms

- Only one way to go from extensive to normal form but potentially several ways to go from normal to extensive form, as the following example indicates.





		Player 2	
		C	D
Player 1	A	1,2	1,2
	B	3,1	2,4

- For this reason, we have to accurately describe which game we have in mind (the game tree in the left or right panel).

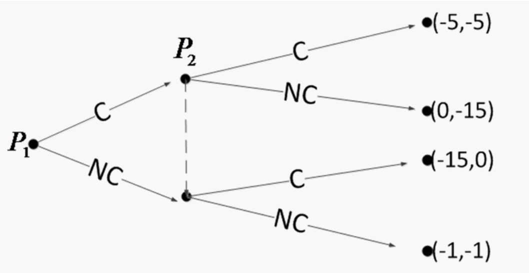
- **Additional practice?**
  - See "Guided Exercise" in page 34 of Watson.
  - This exercise transforms the Katsenberg-Eisner game into its matrix (normal form) representation.

von Neumann-Morgenstern expected utility function (WATSON CH.4)

- Expected utility (EU) that player i obtains from playing strategy s_i :

$$EU\left(s_i\right) = p_1 \cdot u\left(I_1\right) + p_2 \cdot u\left(I_2\right) + \ldots$$

  - Example:



- Let's consider that player 1 in the above game has a Bernoulli's utility function given by $u(I) = 3 \cdot I$,where I denotes income.
- Then, player 1 obtains the following expected utility from selecting C,

$$\begin{aligned} EU_1(C) &= prob(C) \cdot u((C,C)) + prob(NC) \cdot u((C,NC)) \\ &= p \cdot 3 \cdot (-5) + (1-p) \cdot 0 = -15p \end{aligned}$$

  where p represents the probability that player 2 chooses C.
- Similarly, player 1's expected utility from selecting NC is

$$\begin{aligned} EU_1(NC) &= prob(C) \cdot u((NC,C)) + prob(NC) \cdot u((NC,NC)) \\ &= p \cdot 3 \times (-15) + (1-p) \cdot (3 \times (-1)) = 3 - 42p \end{aligned}$$

- In order to challenge ourselves a little bit further, let's find the expected utility that player 1 in the following game obtains when selecting U , C or D...
    - assuming that the probability with which his opponet, player 2, selects L, M and R are 1/2, 1/4 and 1/4 respectively.

		PLAYER 2		
		$\frac{1}{2}$L	$\frac{1}{4}$M	$\frac{1}{4}$R
PLAYER 1	U	8,1	0,2	4,0
	C	3,3	1,2	0,0
	D	5,0	2,3	8,1

- If player 1 believes player 2 will randomize according to probability distribution $\theta_2 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$,then player 1's expected utility is:

$$\begin{aligned}
EU_1\left(U, \theta_2\right) &= \frac{1}{2} \cdot 8 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 4 = 5 \\
EU_1\left(C, \theta_2\right) &= \frac{1}{2} \cdot 3 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 0 = \frac{7}{4} \\
EU_1\left(D, \theta_2\right) &= \frac{1}{2} \cdot 5 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 8 = 5
\end{aligned}$$

- What if player 2 believes player 1 will select $\theta_1 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ (U,C,D), and player 2 himself plans to randomize using $\left(0, \frac{1}{2}, \frac{1}{2}\right)$?
- Try on your own (answer in guided exercise, Ch4 Watson)

- **We are done discribing games!!**
  - We will return to some additional properties of game trees later on, but only for a second.
- **Let's start solving games!!**
  - We will use solution concepts that will help us predict the precise strategy that every player selects in the game.
- **Our goal:**
  - To be as precise as possible in our equilibrium predictions.
  - Hence, we will present (and rank) solution concepts in terms of their predictive power.

Best Response

- Given the previous three problems when applying dominated strategies, let's examine another solution concept:
    - Using Best responses to find Rationalizable strategies, and Nash equilibria.

Best Response

- **Best response:**
  - A strategy s_i^* is a best response of player i to a strategy profile s_{-i} selected by all other players if it provides player i a larger payoff than any of his available strategies $s_i \in S_i$.

    $$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_i \in S_i$$

  - For two players, s_1^* is a best response to a strategy s_2 selected by player 2 if

    $$u_1(s_1^*, s_2) \geq u_1(s_1, s_2) \text{ for all } s_1 \in S_1$$

    That is, when player 2 selects s_2, the utility player 1 obtains from playing s_1^* is higher than by playing any other of his available strategies.

Rationalizable strategies

- Given the definition of a best response for player i, we can interpret that he will never use a strategy that cannot be rationalized for any beliefs about his opponents' strategies:
  - A strategy $s_i \in S_i$ is **never a best response** for player i if there are no beliefs he can sustain about the strategies that his opponents will select, s_{-i}, for which s_i is a best response.
  - We can then eliminate strategies that are never a best response from S_i, as they are not rationalizable.
- In fact, the only strategies that are rationalizable are those that survive such iterative deletion, as we define next:
  - A strategy profile $\left(s_1^*, s_2^*, ..., s_N^*\right)$ is **rationalizable** if it survives the iterative elimination of those strategies that are never a best response.
- Examples, and comparison with IDSDS (see Handout).

Rationalizable Strategies - Example

1. Beauty Contest / Guess the Average $[0, 100]$

The guess which is closest to $\frac{1}{2}$ the average wins a prize.

"Level 0"Players$\longrightarrow$They select a random number from $[0, 100]$, implying an average of 50.

$$\begin{array}{rcl} \text{"Level 1" Players} & \longrightarrow & BR(s_{-i}) = BR(50) = 25 \\ \text{"Level 2" Players} & \longrightarrow & BR(s_{-1}) = BR(25) = 12.5 \\ \dots & \longrightarrow & 0 \end{array}$$

Rationalizable Strategies

How many degrees of iteration do subjects use in experimental settings?

- About 1-2 for "regular" people.
  - So they say $s_i = 50$ or $s_i = 25$.
- But...
  - One step more for undergrads who took game theory;
  - One step more for Portfolio managers;
  - 1-2 steps more for Caltech Econ majors;
  - About 3 more for usual readers of financial newspapers (*Expansión* in Spain and *FT* in the UK).

For more details, see Rosemarie Nagel "Unraveling in Guessing Games: An Experimental Study" (1995). *American Economic Review*, pp. 1313-26.

Nash equilibrium

- Besides rationalizability, we can use best responses to identify the Nash equilibria of a game, as we do next.

Nash equilibrium

- A strategy profile $(s_1^*, s_2^*, ..., s_N^*)$ is a Nash equilibrium if every player's strategy is a best response to his opponent's strategies, i.e., if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \text{ for all } s_i \in S_i \text{ and for every player } i$$

- For two players, a strategy pair (s_1^*, s_2^*) is a Nash equilibrium if
  - Player 1's strategy, s_1^*, is a best response to player 2's strategy s_2^*,

$$u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*) \text{ for all } s_1 \in S_1 \implies BR_1(s_2^*) = s_1^*$$

  - and similarly, player 2's strategy, s_2^*, is a best response to player 1's strategy s_1^*,

$$u_2(s_1^*, s_2^*) \geq u_2(s_1, s_2^*) forall s_2 \in S_2 \implies BR_2(s_1^*) = s_2^*$$

Nash equilibrium

- In short, every player must be playing a best response against his opponent's strategies, and
- Players' conjectures must be correct in equilibrium
  - Otherwise, players would have incentives to modify their strategy.
  - This didn't need to be true in the definition of Rationalizability, where beliefs could be incorrect.
- The Nash equilibrium strategies are stable, since players don't have incentives to deviate.

Nash equilibrium

- *Note:*
    - While we have described the concept of best response and Nash equilibrium for the case of pure strategies (no randomizations), our definitions and examples can be extended to mixed strategies too.
    - We will next go over several examples of pure strategy Nash equilibria (psNE) and afterwards examine mixed strategy Nash equilibria (msNE).

Example 1: Prisoner's Dilemma

If Player 2 confesses,
$BR_1(C)=C$

		Player 2	
		Confess	Not Confess
Player 1	Confess	$\underline{-5}$,-5	0,-15
	Not Confess	-15,0	-1,-1

- Let's start analyzing player 1's best responses.
- If player 2 selects Confess (left column), then player 1's best response is to confess as well.
- For compactness, we represent this result as $BR_1(C) = C$, and underline the payoff that player 1 would obtain after selecting his best response in this setting, i.e., -5.

Example 1: Prisoner's Dilemma

If Player 2 does not confess,
$BR_1(NC)=C$

		Player 2	
		Confess	Not Confess
Player 1	Confess	-5,-5	$\underline{0}$,-15
	Not Confess	-15,0	-1,-1

- Let's continue analyzing player 1's best responses.
- If player 2 selects, instead, Not Confess (right column), then player 1's best response is to confess.
- For compactness, we represent this result as $BR_1(NC) = C$, and underline the payoff that player 1 would obtain after selecting his best response in this setting, i.e., 0.

Example 1: Prisoner's Dilemma

If Player 1 confesses,
$BR_2(C)=C$

		Player 2	
		Confess	Not Confess
Player 1	Confess	-5,$\underline{-5}$	0,-15
	Not Confess	-15,0	-1,-1

- Let's now move to player 2's best responses.
- If player 1 selects Confess (upper row), then player 2's best response is to confess.
- For compactness, we represent $BR_2(C) = C$, and underline the payoff that player 2 would obtain after selecting his best response in this setting, i.e., -5.

Example 1: Prisoner's Dilemma

If Player 1 does not confess,
$BR_2(NC)=C$

		Player 2	
		Confess	Not Confess
Player 1	Confess	-5,-5	0,-15
	Not Confess	-15,$\underline{0}$	-1,-1

- Finally, if player 1 selects Not Confess (lower row), then player 2's best response is to confess.
- For compactness, we represent $BR_2(NC) = C$, and underline the payoff that player 2 would obtain after selecting his best response in this setting, i.e., 0.

Example 1: Prisoner's Dilemma

- Underlined payoffs hence represent the payoffs that players obtain when playing their best responses.
- When we put all underlined payoffs together in the prisoner's dilemma game...

		Player 2	
		Confess	Not Confess
Player 1	Confess	$\underline{-5}$,$\underline{-5}$	$\underline{0}$,-15
	Not Confess	-15,$\underline{0}$	-1,-1

- We see that there is only one cell where the payoffs of both player 1 and 2 were underlined.
- In this cell, players must be selecting mutual best responses, implying that this cell is a Nash equilibrium of the game.
- Hence, we say that the NE of this game is (Confess, Confess) with a corresponding equilibrium payoff of $(-5, -5)$.

Example 2: Battle of the Sexes

- Recall that this is an example of a coordination game, such as those describing technology adoption by two firms.

		Wife	
		Football	Opera
Husband	Football	$\underline{3}, 1$	$0, 0$
	Opera	$0, 0$	$\underline{1}, 3$

- **Husband's best responses:**
  - When his wife selects the Football game, his best response is to also go to the Football game, i.e., $BR_H(F) = F$.
  - When his wife selects Opera, his best response is to also go to the Opera, i.e., $BR_H(O) = O$.

Example 2: Battle of the Sexes

		Wife	
		Football	Opera
Husband	Football	$3, \underline{1}$	$0, 0$
	Opera	$0, 0$	$1, \underline{3}$

- **Wife's best responses:**
  - When her husband selects the Football game, her best response is to also go to the Football game, i.e., $BR_W(F) = F$.
  - When her husband selects Opera, her best response is to also go to the Opera, i.e., $BR_W(O) = O$.

Example 2: Battle of the Sexes

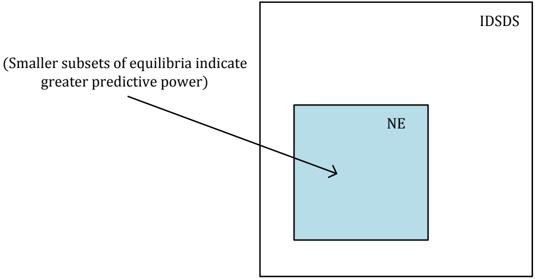
		Wife	
		Football	Opera
Husband	Football	$\underline{3}, \underline{1}$	$0, 0$
	Opera	$0, 0$	$\underline{1}, \underline{3}$

- Two cells have all payoffs underlined. These are the two Nash equilibria of this game:
  - (Football, Football) with equilibrium payoff $(3, 1)$, and
  - (Opera, Opera) with equilibrium payoff $(1, 3)$.

- Prisoner's Dilemma $\longrightarrow$ NE $=$ set of strategies surviving IDSDS
- Battle of the Sexes $\longrightarrow$ NE is a subset of strategies surviving IDSDS (the entire game).

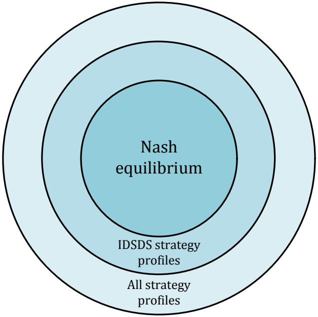
Therefore, NE has more predictive power than IDSDS.

- Great!



(Smaller subsets of equilibria indicate greater predictive power)

The NE provides more precise equilibrium predictions:



Hence, if a strategy profile (s_1^*, s_2^*) is a NE, it must survive IDSDS. However, if a strategy profile (s_1^*, s_2^*) survives IDSDS, it does not need to be a NE.

Example 3: Pareto coordination

		Player 2	
		Tech A	Tech B
Player 1	Tech A	$\underline{2}, \underline{2}$	0, 0
	Tech B	0, 0	$\underline{1}, \underline{1}$

- While we can find two NE in this game,(A,A) and (B,B), there are four strategy profiles surviving IDSDS
    - Indeed, since no player has strictly dominated strategies, all columns and rows survive the application of IDSDS.

Example 3: Pareto coordination

		Player 2	
		Tech A	Tech B
Player 1	Tech A	$\underline{2}, \underline{2}$	0, 0
	Tech B	0, 0	$\underline{1}, \underline{1}$

- While two NE can be sustained, (B,B) yields a lower payoff than (A,A) for both players.
- Equilibrium (B,B) occurs because, once a player chooses B, his opponent is better off at B than at A.
- In other words, they would have to sumultaneously move to A in order to increase their payoffs.

Example 3: Pareto coordination

- Such a miscoordination into the "bad equilibrium" (B,B) is more recurrent than we think:
  - Betamax vs. VHS (where VHS plays the role of the inferior technology B, and Betamax that of the superior technology A). Indeed, once all your friends have VHS, your best response is to buy a VHS as well.
  - Mac vs. PC (before files were mostly compatible).
  - Blu-ray vs. HD-DVD.

Example 4: Anticoordination Game

- The game of chicken is an example of an anticoordination game.

		Dean	
		Swerve	Straight
James	Swerve	0, 0	$\underline{-1}$, 1
	Straight	$\underline{1}$, -1	-2, -2

- **James' best responses:**
  - When Dean selects Swerve, James' best response is to drive Straight, i.e., $BR_J(Swerve) = Straight$.
  - When Dean selects Straight, James' best response is to Swerve, i.e., $BR_J(Straight) = Swerve$.

Example 4: Anticoordination Game

		Dean	
		Swerve	Straight
James	Swerve	0, 0	-1, $\underline{1}$
	Straight	1, $\underline{-1}$	-2, -2

- **Dean's best responses:**
  - When James selects Swerve, Dean's best response is to drive Straight, i.e., $BR_D(Swerve) = Straight$.
  - When James selects Straight, Dean's best response is to Swerve, i.e., $BR_D(Straight) = Swerve$.

Example 4: Anticoordination Game

		Dean	
		Swerve	Straight
James	Swerve	0, 0	$\underline{-1}, \underline{1}$
	Straight	$\underline{1}, \underline{-1}$	-2, -2

- Two cells have all payoffs underlined. These are the two NE of this game:
  - (Swerve, Straight) with equilibrium payoff (-1,1), and
  - (Straight, Swerve) with equilibrium payoff (1,-1).
- Unline in coordination games, such as the Battle of the Sexes or technology games, here every player seeks to choose the opposite strategy of his opponent.

Some Questions about NE:

1. Existence? $\longrightarrow$ all the games analyzed in this course will have at least one NE (in pure or **mixed** strategies)
2. Uniqueness? $\longrightarrow$ Small predictive power. Later on we will learn how to restrict the set of NE.

Example 6: Rock-Paper-Scissors

- Not all games must have one NE using pure strategies...

			Lisa	
		Rock	Paper	Scissors
Bart	Rock	0, 0	-1, 1	$\underline{1}$, -1
	Paper	$\underline{1}$, -1	0, 0	-1, 1
	Scissors	-1, 1	$\underline{1}$, -1	0, 0

- **Bart's best responses:**
  - If Lisa chooses Rock, then Bart's best response is to choose Paper, i.e., $BR_B(Rock) = Paper$.
  - If Lisa chooses Paper, then Bart's best response is to choose Scissors, i.e., $BR_B(Paper) = Scissors$.
  - If Lisa chooses Scissors, then Bart's best response is to choose Rock, i.e., $BR_B(Scissors) = Rock$.

Example 6: Rock-Paper-Scissors

			Lisa	
		Rock	Paper	Scissors
	Rock	0, 0	-1, $\underline{1}$	$\underline{1}$, -1
Bart	Paper	1, -1	0, 0	-1, $\underline{1}$
	Scissors	-1, $\underline{1}$	$\underline{1}$, -1	0, 0

- **Lisa's best responses:**
    - If Bart chooses Rock, then Lisa's best response is to choose Paper, i.e., $BR_L(Rock) = Paper$.
    - If Bart chooses Paper, then Lisa's best response is to choose Scissors, i.e., $BR_L(Paper) = Scissors$.
    - If Bart chooses Scissors, then Lisa's best response is to choose Rock, i.e., $BR_L(Scissors) = Rock$.

Example 6: Rock-Paper-Scissors

		Lisa		
		Rock	Paper	Scissors
	Rock	0, 0	-1, $\underline{1}$	$\underline{1}$, -1
Bart	Paper	$\underline{1}$, -1	0, 0	-1, $\underline{1}$
	Scissors	-1, $\underline{1}$	$\underline{1}$, -1	0, 0

- In this game, there are no NE using pure strategies!
  - But it will have a NE using mixed strategies (In a couple of weeks).

Example 7: Game with Many Strategies

		Player 2			
		w	x	y	z
Player 1	a	0, 1	0, 1	1, 0	$\underline{3}$, 2
	b	1, 2	$\underline{2}$, 2	$\underline{4}$, 0	0, 2
	c	2, 1	0, 1	1, 2	1, 0
	d	$\underline{3}$, 0	1, 0	1, 1	$\underline{3}$,1

- **Player 1's best responses:**
  - If Player 2 chooses w, then Player 1's best response is to choose d, i.e., $BR_1(w) = d$.
  - If Player 2 chooses x, then Player 1's best response is to choose b, i.e., $BR_1(x) = b$.
  - If Player 2 chooses y, then Player 1's best response is to choose b, i.e., $BR_1(y) = b$.
  - If Player 2 chooses z, then Player 1's best response is to choose a or d, i.e., $BR_1(z) = \{a, d\}$.

Example 7: Game with Many Strategies

		Player 2			
		w	x	y	z
Player 1	a	0, 1	0, 1	1, 0	3, $\underline{2}$
	b	1, $\underline{2}$	2, $\underline{2}$	4, 0	0, $\underline{2}$
	c	2, 1	0, 1	1, $\underline{2}$	1, 0
	d	3, 0	1, 0	1, $\underline{1}$	3,$\underline{1}$

- **Player 2's best responses:**
  - If Player 1 chooses a, then Player 2's best response is to choose z, i.e., $BR_1(a) = z$.
  - If Player 1 chooses b, then Player 2's best response is to choose w, x or z, i.e., $BR_1(b) = \{w, x, z\}$.
  - If Player 1 chooses c, then Player 2's best response is to choose y, i.e., $BR_1(c) = y$.
  - If Player 1 chooses d, then Player 2's best response is to choose y or z, i.e., $BR_1(d) = \{y, z\}$.

Example 7: Game with Many Strategies

		Player 2			
		w	x	y	z
Player 1	a	0, 1	0, 1	1, 0	$\underline{3}$, $\underline{2}$
	b	1, $\underline{2}$	$\underline{2}$, $\underline{2}$	$\underline{4}$, 0	0, $\underline{2}$
	c	2, 1	0, 1	1, $\underline{2}$	1, 0
	d	$\underline{3}$, 0	1, 0	1, $\underline{1}$	$\underline{3}$,$\underline{1}$

- NE can be applied very easily to games with many strategies. In this case, there are 3 seperate NE: (b,x), (a,z) and (d,z).
- Two important points:
    - Note that BR cannot be empty: I might be indifferent among my available strategies, but BR is non-empty.
    - Another important point: Players can use weakly dominated strategies, i.e., a or d by Player 1; y or z by Player 2.

Example 8: The American Idol Fandom

- We can also find the NE in 3-player games.
  - Harrington, pp. 101-102.
  - More generally representing a coordination game between three individuals or firms.
- "Alicia, Kaitlyn, and Lauren are ecstatic. They've just landed tickets to attend this week's segment of American Idol. The three teens have the same favorite among the nine contestants that remain: Ace Young. They're determined to take this opportunity to make a statement. While [text]ing, they come up with a plan to wear T-shirts that spell out "ACE" in large letters. Lauren is to wear a T-shirt with a big "A," Kaitlyn with a "C," and Alicia with an "E." If they pull this stunt off, who knows–they might end up on national television! OMG!

Example 8: The American Idol Fandom

- While they all like this idea, each is tempted to wear instead an attractive new top just purchased from their latest shopping expedition to Bebe. It's now an hour before they have to leave to meet at the studio, and each is at home trying to decide between the Bebe top and the lettered T-shirt. What should each wear?"

Alicia chooses E

		Kaitlyn	
		C	Bebe
Lauren	A	2, 2, 2	0, 1, 0
	Bebe	1, 0, 0	1, 1, 0

Alicia chooses Bebe

		Kaitlyn	
		C	Bebe
Lauren	A	0, 0, 1	0, 1, 1
	Bebe	1, 0, 1	1, 1, 1

Example 8: The American Idol Fandom

Alicia chooses E

		Kaitlyn	
		C	Bebe
Lauren	A	$\underline{2}, \underline{2}, \underline{2}$	0, 1, 0
	Bebe	1, 0, 0	$\underline{1}, \underline{1}, 0$

Alicia chooses Bebe

		Kaitlyn	
		C	Bebe
Lauren	A	0, 0, 1	$0, \underline{1}, \underline{1}$
	Bebe	$\underline{1}, 0, \underline{1}$	$\underline{1}, \underline{1}, \underline{1}$

- There are 2 psNE: (A,C,E) and (Bebe, Bebe, Bebe)

Games with Continuous Actions Spaces

- So far, we considered that players select one among a discrete list of available actions, e.g., $s_i \in \{Enter, NotEnter\}$, $s_i \in \{x, y, z\}$.
- But in some economic settings, agents can select among an infinite list of actions.
  - **Examples:** an output level $q_i \in \mathbb{R}_+$ (as in the Cournot game of output competition),
  - A price level $p_i \in \mathbb{R}_+$ (as in the Bertrand game of price competition),
  - Contribution $c_i \in \mathbb{R}_+$ to a charity in a public good game,
  - Exploitation level $x_i \in \mathbb{R}_+$ of a common pool resource, etc.

Cournot Game of Output Competition

- We first assume that $N = 2$ firms compete selling a homogenous product (no product differentiation).
  - Later on (maybe in a homework) you will analyze the case where firms sell differentiated products (easy! don't worry).
- Firm i's total cost function is $TC_i(q_i) = c_i q_i$.
  - Note that this allows for firms to be symmetric in costs, $c_i = c_j$, or asymmetric, $c_i > c_j$.
- Inverse demand function is linear $p(Q) = a - bQ$, where $Q = q_1 + q_2$ denotes the aggregate output, $a > c$ and $b > 0$.

Cournot Game of Output Competition

- Since $p(Q) = a - bQ$, where $Q = q_1 + q_2$, the profit maximization problem for firm 1 is therefore

$$\begin{aligned} \max_{q_1} \quad \pi_1(q_1, q_2) &= [a - b(q_1 + q_2)]q_1 - c_1 q_1 \\ &= aq_1 - b(q_1 + q_2)q_1 - c_1 q_1 \\ &= aq_1 - bq_1^2 - bq_1 q_2 - c_1 q_1 \end{aligned}$$

Cournot Game of Output Competition

- Taking first-order conditions with respect to q_1,

$$a - 2bq_1 - bq_2 - c_1 = 0$$

and solving for q_1, we obtain

$$q_1 = \frac{a - c_1}{2b} - \frac{1}{2}q_2$$

Cournot Game of Output Competition

- Using $q_1 = \frac{a-c_1}{2b} - \frac{1}{2}q_2$, note that:
  - q_1 is positive when $q_2 = 0$, i.e., $q_1 = \frac{a-c_1}{2b}$, but...
  - q_1 decreases in q_2, becoming zero when q_2 is sufficiently large. In particular, $q_1 = 0$, when

$$0 = \frac{a - c_1}{2b} - \frac{1}{2}q_2 \implies \frac{a - c_1}{b} = q_2$$

Cournot Game of Output Competition

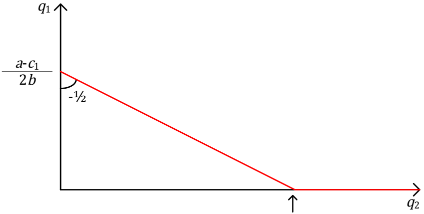
- We can hence, report firm 1's profit maximizing output as follows

$$q_1(q_2) = \begin{cases} \frac{a-c_1}{2b} - \frac{1}{2}q_2 \text{ if } q_2 \leq \frac{a-c_1}{b} \\ 0 \qquad\qquad\quad \text{ if } q_2 > \frac{a-c_1}{b} \end{cases}$$

- This is firm 1's **best response function**: it tells firm 1 how many units to produce in order to maximize profits as a function of firm 2's output, q_2 [See figure].

Cournot Game of Output Competition

- Drawing a single BRF: $q_1(q_2) = \begin{cases} \frac{a-c_1}{2b} - \frac{1}{2}q_2 & \text{if } q_2 \leq \frac{a-c_1}{b} \\ 0 & \text{if } q_2 > \frac{a-c_1}{b} \end{cases}$



At this point, $q_1 = 0 = \frac{a\text{-}c_1}{2b} - \frac{1}{2}q_2$.

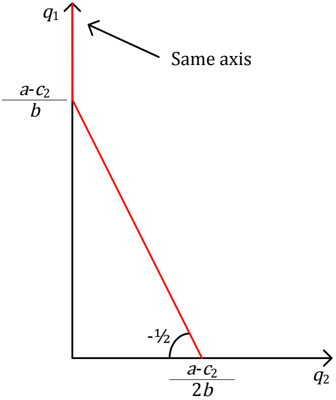
- In order to find the horizontal intercept, where $q_1 = 0$, we solve for q_2, as follows

$$0 = \frac{a-c_1}{2b} - \frac{1}{2}q_2 \implies \frac{a-c_1}{b} = q_2$$

- Hence, the horizontal intercept of BRF_1 is $q_2 = \frac{a-c_1}{b}$

Cournot Game of Output Competition

- Similarly for BRF_2: $q_2(q_1) = \begin{cases} \frac{a-c_2}{2b} - \frac{1}{2}q_2 & \text{if } q_1 \leq \frac{a-c_2}{b} \\ 0 & \text{if } q_1 > \frac{a-c_2}{b} \end{cases}$
- Note that we depict BRF_2 using the same axis as for BRF_1 in order to superimpose both BRFs later on.

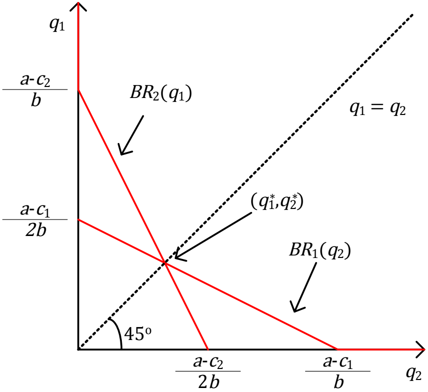


Cournot Game of Output Competition

- Putting both firms' *BRF* together... we obtain two figures:
    - one for the case in which firms are symmetric in marginal costs, $c_1 = c_2$, and
    - another figure for the case in which firms are asymmetric, $c_2 > c_1$.

Cournot Game of Output Competition

- If $c_1 = c_2$, (firms are symmetric in costs),



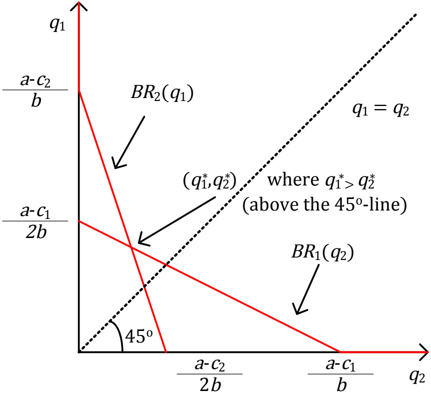
Cournot Game of Output Competition

- Since $c_1 = c_2$, then

$$\begin{aligned} \frac{a - c_1}{2b} &= \frac{a - c_2}{2b} \text{ (vertical intercepts)} \\ \frac{a - c_1}{b} &= \frac{a - c_2}{b} \text{ (horizontal intercepts)} \end{aligned}$$

Cournot Game of Output Competition

- If $c_2 > c_1$ (firm 1 is more competitive),



Cournot Game of Output Competition

- Since $c_2 > c_1$,

$$\begin{aligned} \frac{a-c_1}{2b} &> \frac{a-c_2}{2b} \text{ (vertical intercepts)} \\ \frac{a-c_1}{b} &> \frac{a-c_2}{b} \text{ (horizontal intercepts)} \end{aligned}$$

Cournot Game of Output Competition

- How can we find the NE of this game?
  - We know that each firm must be using its BRF in equilibrium.
  - We must then find the point where BRF_1 and BRF_2 cross each other.
  - Assuming an interior solution,

$$BRF_1 \longrightarrow q_1 = \frac{a-c_1}{2b} - \frac{1}{2}q_2 = \frac{a-c_1}{2b} - \frac{1}{2}\left(\underbrace{\frac{a-c_2}{2b} - \frac{1}{2}q_1}_{BRF_2}\right)$$

and solving for q_1,

$$q_1 = \frac{a-2c_1+c_2}{3b}$$

Similarly for q_2,

$$q_2 = \frac{a-2c_2+c_1}{3b}$$

Cournot Game of Output Competition

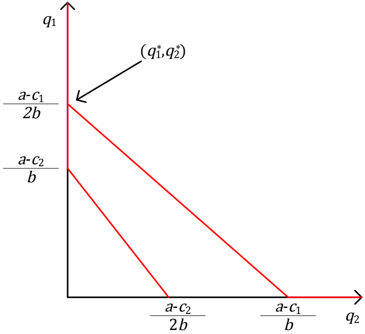
- What about Corner Solutions?
  - Using the figures, we can easily determine a condition for firm 2's equilibrium output, q_2^*, to be zero...
  - In particular, the horizontal intercept of firm 2's *BRF* lies below the vertical intercept of firm 1's *BRF*.
    - That is, if

$$\frac{a-c_2}{b} < \frac{a-c_1}{2b} \iff \frac{a+c_1}{2} < c_2$$

- As depicted in the next figure

Cournot Game of Output Competition

- Corner Solution with only firm 1 producing



- Note that (q_1^*, q_2^*) is the only crossing point between BRF_1 and BRF_2, implying $q_1^* > 0$, but $q_2^* = 0$.

Cournot Game of Output Competition

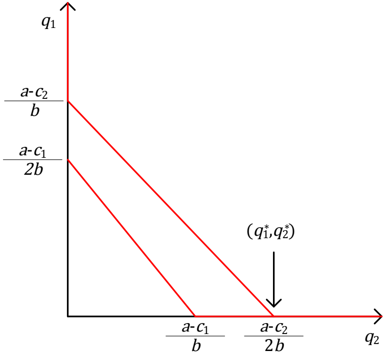
- This corner solution happens when

$$\frac{a - c_2}{b} < \frac{a - c_1}{2b} \iff \frac{a + c_1}{2} < c_2$$

- **Intuition:** Firm 1 is super-competitive (High c_2).

Cournot Game of Output Competition

- Another Corner Solution with only firm 2 producing:



- Note that (q_1^*, q_2^*) is the only crossing point between BRF_1 and BRF_2, implying $q_2^* > 0$, but $q_1^* = 0$.

Cournot Game of Output Competition

- This corner solution happens when

$$\frac{a-c_2}{b} > \frac{a-c_1}{2b} \iff \frac{a+c_1}{2} > c_2$$

- **Intuition:** Firm 2 is super-competitive (Low c_2).

Cournot Game of Output Competition

- Hence, aggregate output (assuming interior solutions) is

$$Q = q_1 + q_2 = \frac{a - 2c_1 + c_2}{3b} + \frac{a - 2c_2 + c_1}{3b} = \frac{2a - c_1 - c_2}{3b}$$

  and the equilibrium price is

$$p = a - bQ = a - b\left(\underbrace{\frac{2a - c_1 - c_2}{3b}}_{Q}\right) = \frac{a + c_1 + c_2}{3}.$$

- Assuming symmetry $(c_1 = c_2 = c)$, profits are

$$\pi_i = (p - c)q_i = \left(\frac{a + 2c}{3} - c\right)\frac{a - c}{3b} = \frac{(a - c)^2}{9b}$$

- **Practice**: find profits *without symmetry*. If we assume that $c_2 > c_1$, which firm experiences the highest profit?

Cournot Game of Output Competition

- This is very similar to the prisoner's dilemma!
- Indeed, if firms coordinate their production to lower production levels, they would maximize their joint profits.
    - Let us show how (for simplicity we assume symmetry in costs).
- First, note that firms would maximize their joint profits by choosing q_1 and q_2 such that

$$\begin{aligned} \max \quad \pi_1 + \pi_2 &= [(a - b(q_1 + q_2))q_1 - cq_1] \\ &\quad +[(a - b(q_1 + q_2))q_2 - cq_2] \\ &= (a - bQ)Q - cQ \\ &= aQ - bQ^2 - cQ \end{aligned}$$

Cournot Game of Output Competition

- Taking first-order conditions with respect to Q, we obtain

  $$a - 2bQ - c = 0$$

  and solving for Q,we obtain the aggregate output level for the cartel

  $$Q = \frac{a-c}{2b}$$

- Since firms are symmetric in costs, each produces half of this aggregate output level,

  $$q_i = \frac{1}{2}\frac{a-c}{2b}$$

Cournot Game of Output Competition

Hence, equilibrium price is

$$p = a - bQ = a - b\left(\frac{a-c}{2b}\right) = \frac{a+c}{2}$$

and profits for every firm i are

$$\pi_i = p \cdot q_i - cq_i = \frac{a+c}{2}\left(\frac{a-c}{2b}\right) - c\left(\frac{a-c}{4b}\right) = \frac{(a-c)^2}{8b}$$

which is higher than the individual profit for every firm under Cournot competition, $\frac{(a-c)^2}{9b}$.

Cournot Game of Output Competition

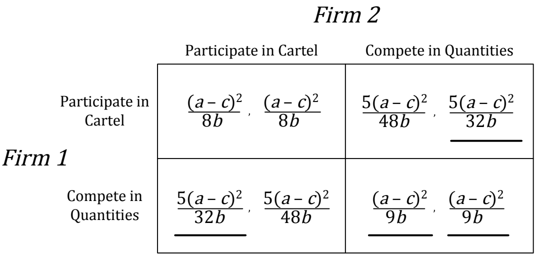
- What if my firm deviates to Cournot output?

$$\begin{aligned}\pi_i &= pq_i - cq_i = \left[a - b\left(\underbrace{\frac{a-c}{3b}}_{q_i^{Cournot}} + \underbrace{\frac{a-c}{4b}}_{q_j^{Cartel}}\right)\right]\cdot\frac{a-c}{3b} \\ &\quad -c\left(\frac{a-c}{3b}\right) \\ &= \frac{5(a-c)^2}{36b}\end{aligned}$$

(and Firm j makes a profit of $\frac{5(a-c)^2}{48b}$).

Cournot Game of Output Competition

- Putting everything together:



		Firm 2	
		Participate in Cartel	Compete in Quantities
Firm 1	Participate in Cartel	$\frac{(a-c)^2}{8b}, \frac{(a-c)^2}{8b}$	$\frac{5(a-c)^2}{48b}, \underline{\frac{5(a-c)^2}{32b}}$
	Compete in Quantities	$\underline{\frac{5(a-c)^2}{32b}}, \frac{5(a-c)^2}{48b}$	$\underline{\frac{(a-c)^2}{9b}}, \underline{\frac{(a-c)^2}{9b}}$

- Conditional on firm 2 participating in the cartel, firm 1 compares $\frac{(a-c)^2}{8b} < \frac{5(a-c)^2}{36b} \iff 0.125 < 0.1388$.
- Conditional on firm 2 competing in quantities, firm 1 compares $\frac{5(a-c)^2}{48b} < \frac{(a-c)^2}{9b} \iff 0.1 < 0.111$.
- (And similarly for firm 2).

- Hence, deviating to Cournot output levels is a best response for every firm regardless of whether its rival respects or violates the cartel agreement.
- In other words, deviating to Cournot output levels is a strictly dominant strategy for both firms, and thus constitutes the NE of this game.
- How can firms then collide effectively? By interacting for several periods. (We will come back to collusive practices in future chapters).

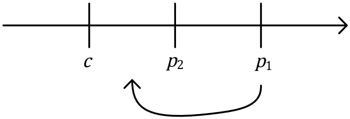
Bertrand Game of Price Competition

Competition in prices. The firm with the lowest price attracts all consumers. If both firms charge the same price, they share consumers equally.

- Any $p_i < c$ is strictly dominated by $p_i \geq c$.
- No *asymmetric* Nash equilibrium: (See Figures)
    1. If $p_1 > p_2 > c$, then firm 1 obtains no profit, and it can undercut firm 2's price to $p_2 > p_1 > c$. Hence, there exists a profitable deviation, which shows that $p_1 > p_2 > c$ cannot be a psNE.
    2. If $p_2 > p_1 > c$. Similarly, firm 2 obtains no profit, but can undercut firm 1's price to $p_1 > p_2 > c$. Hence, there exists a profitable deviation, showing that $p_2 > p_1 > c$ cannot be a psNE.
    3. If $p_1 > p_2 = c$, then firm 2 would want to raise its price (keeping it below p_1). Hence, there is a profitable deviation for firm 2, and $p_1 > p_2 = c$ cannot be a psNE.
    4. Similarly for $p_2 > p_1 = c$.

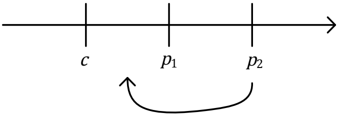
Bertrand Game of Price Competition

1. $p_1 > p_2 > c$



Profitable deviation of firm 1.

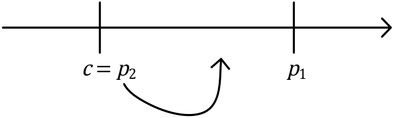
2. $p_2 > p_1 > c$



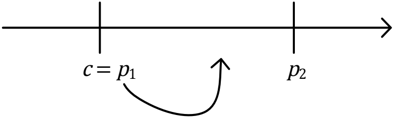
Profitable deviation of firm 2.

Bertrand Game of Price Competition

1. $p_1 > p_2 = c$

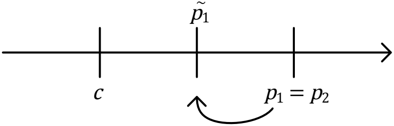


2. $p_2 > p_1 = c$



Bertrand Game of Price Competition

- Therefore, it must be that the psNE is symmetric. If $p_1 = p_2 > c$, then both firms have incentives to deviate, undercutting each other's price (keeping it above c, e.g., $p_2 > \tilde{p}_1 > c$.



And similarly for firm 2

- Hence, $p_1 = p_2 = c$ is the unique psNE.

Bertrand Game of Price Competition

- The Bertrand model of price competition predicts intense competitive pressures until both firms set prices $p_1 = p_2 = c$.
- How can the "super-competitive" outcome where $p_1 = p_2 = c$ be ameliorated? Two ways:
  - Offering price-matching guarantees.
  - Product differentiaion

More Problems that Include Continuum Strategy Spaces

- Let's move outside the realm of industrial organization. There are still several games where players select an action among a continuum of possible actions.
- What's ahead...
- **Tragedy of the commons:** how much effort to exert in fishing, exploiting a forest, etc, incentives to overexploit the resource.
- **Tariff setting by two countries:** what precise tariff to set.
- **Charitable giving:** how many dollars to give to charity.
- **Electoral competition:** political candidates locate their platforms along the line (left-right, more or less spending, more or less security, etc.)
- **Accident law:** how much care a victim and an injurer exert, given different legal rules.

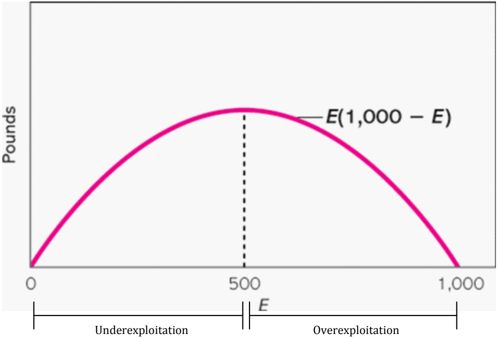
Tragedy of the Commons

- *Reading*: Harrington pp. 164-169.

- n hunters, each deciding how much effort e_i to exert, where

$$e_1 + e_2 + \ldots + e_n = E$$

- Every hunter i's payoff is a function of the total pounds of mammoth killed $Pounds = E(1000 - E)$



Tragedy of the Commons

- From the total pounds of mammoth killed, hunter i obtains a share that depends on how much effort he contributed relative to the entire group, i.e., $\frac{e_i}{E}$.
- Effort, however, is costly for hunter i, at a rate of 100 per unit (opportunity cost of one hour of effort = gathering fruit?).
- Hence, every hunter i's payoff is given by

$$u_i(e_i, e_{-i}) = \underbrace{\frac{e_i}{E}}_{\text{share}} \underbrace{E(1000 - E)}_{\text{total pounds}} - \underbrace{100e_i}_{\text{cost}}$$

cancelling E and rearranging, we obtain

$$e_i \left[1000 - \underbrace{(e_1 + e_2 + \ldots + e_n)}_{E} \right] - 100e_i$$

Tragedy of the Commons

- Taking FOCs with respect to e_i,

$$\frac{\partial u_i(e_i, e_{-i})}{\partial e_i} = 1000 - (e_1 + e_2 + \ldots + e_n) - e_i - 100 = 0$$

and noting that
$e_1 + e_2 + \ldots + e_n = (e_1 + e_2 + e_{i-1} + e_{i+1} + \ldots + e_n) + e_i$,
we can rewrite the above FOC as

$$900 - (e_1 + e_2 + e_{i-1} + e_{i+1} + \ldots + e_n) - 2e_i = 0$$

(SOCs are also satisfied and equal to -2)

- Solving for e_i,

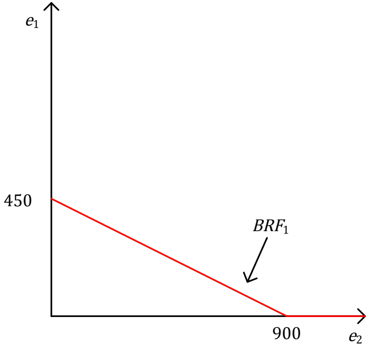
$$e_i = 450 - \frac{e_1 + e_2 + e_{i-1} + e_{i+1} + \ldots + e_n}{2} \qquad (BRF_i)$$

- Intuitively, there exists a strategic substitutability between efforts:
  - the more you hunt, the less prey is left for me.

Tragedy of the Commons

- Note that for the case of only two hunters,

$$e_1 = 450 - \frac{e_2}{2}$$



Tragedy of the Commons

- A similar maximization problem (and resulting *BRF*) can be found for all hunters, since they are all symmetric.
- Hence, $e_1^* = e_2^* = \ldots = e_n^* = e^*$ (symmetric equilibrium) implying that $e_1^* + e_2^* + e_{i-1}^* + e_{i+1}^* + \ldots + e_n^* = (n-1)e^*$.
- Putting this information into the *BRF* yeilds

$$e^* = 450 - \frac{e_1^* + e_2^* + e_{i-1}^* + e_{i+1}^* + \ldots + e_n^*}{2} = 450 - \frac{(n-1)e^*}{2}$$

and solving for e^*, we obtain

$$e^* = \frac{900}{n+1}$$

Tragedy of the Commons

- **Comparative statics on the above result:**
- First, note that individual equilibrium effort, e^*, is decreasing in n since

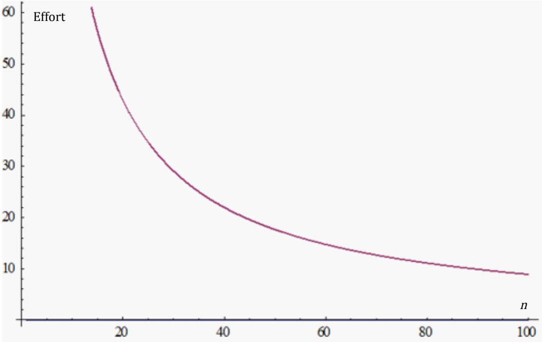
$$\frac{\partial e^*}{\partial n} = -\frac{900}{(n+1)^2} < 0$$

- Intuitively, this implies that an increase in the number of potential hunters reduces every hunter's individual effort, since more hunters are chasing the same set of mammoths. (Why not gather some fruit instead?)

Tragedy of the Commons

- Individual effort in equilibrium

$$e^* = \frac{900}{n+1}$$



Tragedy of the Commons

- Comparative statics on the above result:
- Second, note that aggregate equilibrium effort, ne^*, is increasing in n since

$$\frac{\partial\left(ne^*\right)}{\partial n}=\frac{900(n+1)-900n}{(n+1)^2}=\frac{900}{(n+1)^2}>0$$

- Although each hunter hunts less when there are more hunters, the addition of another hunter offsets that effect, so the total effort put into hunting goes up.

Tragedy of the Commons

- Finally, what about overexploitation?
    - We know that overexploitation occurs if $E > 500$ (the point at which aggregate meat production is maximized).
    - Total effort exceeds 500 if $n\frac{900}{n+1} > 500$, or $n > 1.2$.
    - That is, as long as there are 2 or more hunters, the resource will be overexploited.

Tragedy of the Commons

- The exploitation of a common pool resource (fishing grounds, forests, acquifers, etc.) to a level beyond the level that is socially optimal is referred to as the "**tragedy of the commons**."
    - Why does this "tragedy" occur?
    - Because when an agent exploits the resource he does not take into account the negative effect that his action has on the well-being of other agents exploiting the resource (who now find a more depleted resource).
    - Or more compactly, because every agent does not take into account the negative externality that his actions impose on other agents.