Recitation 1 - EconS 301

1. Consider an individual with Cobb-Douglas utility function

$$u(x, y) = \sqrt{x}\sqrt{y}.$$

Assume that her income is $I = \$120$, the price of good x is $p_x = \$4$, and the price of good y is $p_y = \$10$.

(a) Find the marginal utility of good x, MU_x, and that of good y, MU_y.

- We calculate the marginal utilities by differentiating with respect to each respective variable,

$$MU_x = \frac{\partial u(x,y)}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}} \qquad MU_y = \frac{\partial u(x,y)}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}}$$

(b) Given your results in part (a), does this utility function satisfy monotonicity? And strict monotonicity?

- Since both marginal utilities are strictly positive when the individual consumes a positive amount of both goods, this utility function satisfies both monotonicity and strict monotonicity. If the individual consumes zero units of one (or both) goods, the utility function satisfies monotonicity, but violates strict monotonicity.

(c) Using the marginal utilities you found in part (a), find the marginal rate of substitution of this consumer (MRS).

- The marginal rate of substitution can be found by taking the ratio of our marginal utility with respect to x to our marginal utility with respect to y,

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{\frac{\sqrt{y}}{2\sqrt{x}}}{\frac{\sqrt{x}}{2\sqrt{y}}} = \frac{\sqrt{y}\left(2\sqrt{y}\right)}{\sqrt{x}\left(2\sqrt{x}\right)} = \frac{2y^{\frac{1}{2}+\frac{1}{2}}}{2x^{\frac{1}{2}+\frac{1}{2}}} = \frac{y}{x}$$

(d) Find the equilibrium quantities for goods x and y.

- Tangency condition indicates that

$$\begin{aligned} \frac{y}{x} &= \frac{4}{10} \\ y &= \frac{2}{5}x \end{aligned}$$

and it contains both x and y we move on to step 2a.

- *Step 2a.* Next, we use budget line, $4x + 10y = 120$ and substitute $\frac{2}{5}x$ for y, obtaining

$$4x + 4x = 120.$$

  - Combining terms, we have $8x = 120$, and we can divide both sides of this equation to obtain the amount of good x that she consumes, $x = 15$ units. With a positive value of x, we can move on to step 4.

- *Step 4.* Lastly, we return to our tangency condition to determine how much of good x she consumes.
  - Plugging in our value for x gives $y = \frac{2}{5}(15) = 6$ units.

2. Consider an individual with utility function $u(x, y) = x^2 y$, and facing prices $p_x = \$2$ and $p_y = \$4$.

(a) Assuming that her income is $I = \$800$, find the optimal consumption of goods x and y that maximizes her utility. That is, solve her utility maximization problem (UMP).

- We can derive our demand functions using the same strategies from chapter 3.

- *Step 1.* To solve for our optimal consumption, first we must use the tangency condition, $\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$.
  - For the left-hand side, we calculate the ratio of our marginal utilities, where $MU_x = \frac{\partial u(x,y)}{\partial x} = 2xy$ and $MU_y = \frac{\partial u(x,y)}{\partial y} = x^2$. Therefore,

$$\frac{MU_x}{MU_y} = \frac{2xy}{x^2} = \frac{2y}{x}.$$

  - For the right-hand side, it is simply the ratio of the prices, $\frac{p_x}{p_y}$.
  - Setting them equal to one another gives

$$\frac{2y}{x} = \frac{p_x}{p_y}$$

    which we can rearrange to obtain $x = \frac{2p_y}{p_x}y$. Since our tangency condition contains both x and y we move on to step 2a.
- *Step 2a.* Next, we use our budget line, $p_x x + p_y y = I$ and substitute $\frac{2p_y}{p_x}y$ for x, obtaining

$$p_x \underbrace{\left(\frac{2p_y}{p_x}y\right)}_{x} + p_y y = I.$$

  - Combining terms, we have $3p_y y = I$, and we can divide both sides of this equation to obtain the demand function for good y,

$$y = \frac{I}{3p_y} \text{ units.}$$

    With a positive value of y, we can move on to step 4.
- *Step 4.* Lastly, we return to our tangency condition to determine the demand function for good x.
  - Plugging in our value for y gives

$$x = \frac{2p_y}{p_x}\left(\frac{I}{3p_y}\right) = \frac{2I}{3p_x} \text{ units.}$$

- *Finding initial bundle A.* Plugging in our value for prices, $p_x = \$2$, $p_y = \$4$, and income of $I = \$800$, we have our initial bundle,

$$x = \frac{2(800)}{3(2)} = 266.67 \text{ units} \qquad y = \frac{800}{3(4)} = 66.67 \text{ units.}$$

  This implies that our initial bundle is $A = (266.67, 66.67)$.

(b) Consider now that the price of good y decreases from $p_y = \$4$ to $p'_y = \$3$. Find this consumer's new optimal consumption bundle. Then, identify the total effect of the price change, and decompose it into the substitution and income effects.

- *Finding final bundle C.* Now, we plug in the final prices $p_x = \$2$, $p'_y = \$3$, and income of $I = \$800$ into our demand functions to obtain our final bundle,

$$x = \frac{2(800)}{3(2)} = 266.67 \text{ units} \qquad y = \frac{800}{3(3)} = 88.89 \text{ units.}$$

  This implies that our final bundle is $C = (266.67, 88.89)$.
- *Total Effect.* To find the total effect, we simply take the difference between the final value of good y and the initial value of good y, obtaining,

$$TE = y_C - y_A = 88.89 - 66.67 = 22.22 \text{ units.}$$

- *Finding decomposition bundle B.* Recall that the decomposition bundle must satisfy two conditions:
  - Bundle B must reach the initial utility level. Plugging in our initial bundle A into the utility function provides a utility level of

$$u(266.67, 66.67) = 266.67^2 \times 66.67 = 4,741,096.30.$$

    This implies that our decomposition bundle must satisfy

$$x^2 y = 4,741,096.30.$$

  - Bundle B must be tangent to BL_d, the decomposition budget line which has the same slope as the final budget line. To find this tangency point, we plug in the final prices to our tangency condition from before,

$$\frac{2y}{x} = \frac{p_1}{p_2'} = \frac{2}{3}.$$

    We can rearrange this expression to obtain $x = 3y$.
  - We now have a system of two equations and two unknowns. Substituting $3y$ for x in our utility function, we have

$$\underbrace{(3y)^2}_{x} y = 4,741,096.30$$

    which rearranges to $y^3 = 526,788.48$. Taking the cube root of both sides, we obtain our decomposition value of good y, $y = 80.76$ units. Lastly, we plug our value of y into the decomposition tangency condition to find x,

$$x = 3(80.76) = 242.29 \text{ units}.$$

    This implies that our decomposition bundle is $B = (242.29, 80.76)$.
- *Substitution and Income Effects.* With our decomposition bundle found, we can calculate the substitution effect as the movement along our original utility curve, from bundle A to bundle B,

$$SE = y_B - y_A = 80.76 - 66.67 = 14.09 \text{ units}.$$

  Likewise, our income effect is the movement from our initial utility level to our final utility level, from bundle B to bundle C,

$$IE = y_C - y_B = 88.89 - 80.76 = 8.13 \text{ units}.$$

(c) Considering that the price of good y is still at $p_y = \$4$, assume now that the consumer seeks to reach the same utility level as in part (a). Find the optimal consumption of goods x and y that minimizes her expenditure. That is, solve her expenditure minimization problem (EMP).

- We can derive our demand functions using the same strategies from chapter 3.
- *Step 1.* To solve for our optimal consumption, first we must use the tangency condition, $\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$.
  - For the left-hand side, we calculate the ratio of our marginal utilities, where $MU_x = \frac{\partial u(x,y)}{\partial x} = 2xy$ and $MU_y = \frac{\partial u(x,y)}{\partial y} = x^2$. Therefore,

$$\frac{MU_x}{MU_y} = \frac{2xy}{x^2} = \frac{2y}{x}.$$

  - For the right-hand side, it is simply the ratio of the prices, $\frac{p_x}{p_y}$.
  - Setting them equal to one another gives

$$\frac{2y}{x} = \frac{p_x}{p_y},$$

    which we can rearrange to obtain $x = \frac{2p_y}{p_x} y$. Since our tangency condition contains both x and y we move on to step 2a.

- *Step 2a.* Next, we use our utility curve, $x^2y = u$ and substitute $\frac{2p_y}{p_x}y$ for x, obtaining

$$\underbrace{\left(\frac{2p_y}{p_x}y\right)^2}_{x} y = u.$$

  - Combining terms, we have $\frac{4(p_y)^2}{(p_x)^2}y^3 = u$, and we can divide both sides of this equation, then take a cube root to obtain the compensated demand function for good y,

$$y = \left(\frac{(p_x)^2 u}{4(p_y)^2}\right)^{\frac{1}{3}} \text{ units.}$$

    With a positive value of y, we can move on to step 4.

- *Step 4.* Lastly, we return to our tangency condition to determine the demand function for good x.
  - Plugging in our value for y gives

$$x = \frac{2p_y}{p_x}\left(\frac{(p_x)^2 u}{4(p_y)^2}\right)^{\frac{1}{3}}$$

    Moving all of the terms inside of the exponent, we have,

$$x = \left(\frac{8(p_y)^3}{(p_x)^3}\frac{(p_x)^2 u}{4(p_y)^2}\right)^{\frac{1}{3}}$$

    and simplifying, we obtain,

$$x = \left(\frac{2p_y u}{p_x}\right)^{\frac{1}{3}} \text{ units.}$$

- *Finding initial bundle A.* Plugging in our value for prices, $p_x = \$2$, $p_y = \$4$, and desired utility level of $u = 4,741,096.30$ (from part (b)), we have our initial bundle,

$$\begin{aligned} x &= \left(\frac{2p_y u}{p_x}\right)^{\frac{1}{3}} = \left(\frac{2(4)(4,741,096.30)}{2}\right)^{\frac{1}{3}} = 266.67 \text{ units} \\ y &= \left(\frac{(p_x)^2 u}{4(p_y)^2}\right)^{\frac{1}{3}} = \left(\frac{2^2(4,741,096.30)}{4(4^2)}\right)^{\frac{1}{3}} = 66.67 \text{ units.} \end{aligned}$$

  This implies that our initial bundle is $A = (266.67, 66.67)$.

3. Chris has demand for books (b) and other goods (y) that follows a Cobb-Douglas utility function $u(b,y) = y\sqrt{b}$, and an income of $I = \$50$. Find Chris's Compensating Variation if the price of books decreases from $p_b = \$2$ to $p_b' = \$1$.

   - For this demand, the tangency condition $\frac{MU_b}{MU_y} = \frac{p_b}{p_y}$, or $\frac{y}{2\sqrt{b}\sqrt{b}} = \frac{p_b}{p_y}$, which simplifies to $y = 2p_b b$ since $p_y = \$1$. Substituting this into the budget line $p_b b + p_y y = I$ we obtain $p_b b + 2p_b b = 50$, which simplifies to $3p_b b = 50$, and demand for books b is $b = \frac{50}{3p_b}$. Demand for other good y is

$$y = 2p_b \frac{50}{3p_b} = \frac{100}{3} = 33.33.$$

   (a) *Finding initial bundle A.* At the initial price of $p_b = 2$, the demand for books is $b_A = \frac{50}{3\times 2} = 8.33$ books.

   (b) *Finding final bundle C.* At the final price of $p_b' = 1$, the demand for books is $y_A = \frac{50}{3\times 1} = 16.66$ books.

   (c) *Finding the decomposition bundle B.* At the decomposition bundle, the consumer must:

1. Reach the same utility level as with the initial bundle $A = (8.33, 33.33)$. This bundle yields a utility level of

$$u_A = 33.33\sqrt{8.33} = 96.20$$

The decomposition bundle must also yield a utility level of 96.20, which we can write as

$$(y_B)\sqrt{b_B} = 96.20$$

2. The consumer's indifference curve must be tangent to the budget line, $y = 2p_b'b$, or $y = 2b$ since $p_b' = \$1$. Substituting this into the equation above,

$$(2b_B)\sqrt{b_B} = 96.20$$

Simplifying, we obtain $b_B^{3/2} = \frac{96.20}{2} = 48.1$. Solving for b_B, we get

$$b_B = 40.1^{2/3} \simeq 13.22 \text{ units.}$$

We can insert this into the tangency condition to find that

$$y_B = 2b_B = 2 \times 13.22 = 26.44 \text{ units.}$$

The decomposition bundle is, then, $B = (13.22, 26.44)$.

3. *Evaluating the CV.* The CV is given by $CV = I - I_B$, where $I = \$50$ and I_B is

$$I_B = 13.22 + 26.44 = 39.66$$

The CV is

$$CV = I - I_B = 50 - 39.66 = 10.34$$

After the price decrease, we decrease Chris's income by \$10.34 and his utility level coincides with that before the price decrease.