Advanced Microeconomic Theory

Chapter 5: Choices under Uncertainty

Outline

- Simple, Compound, and Reduced Lotteries
- Independence Axiom
- Expected Utility Theory
- Money Lotteries
- Risk Aversion
- Prospect Theory and Reference-Dependent Utility
- Comparison of Payoff Distributions

Simple, Compound, and Reduced Lotteries

Simple Lotteries

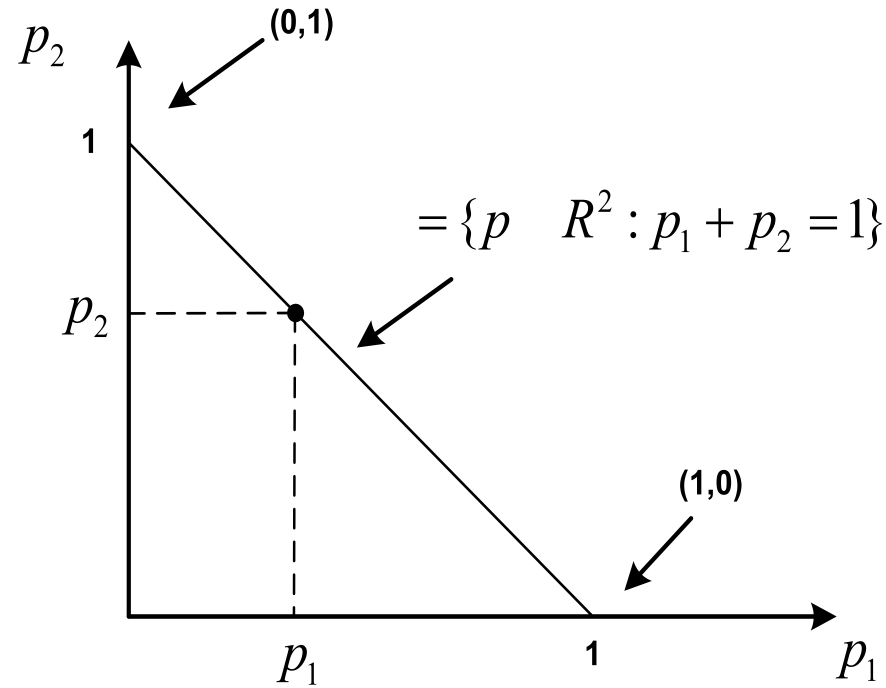
- Consider a set of possible outcomes (or consequences) C.
- The set C can include
  - simple payoffs $C \in \mathbb{R}$ (positive or negative)
  - consumption bundles $C \in \mathbb{R}^L$
- Outcomes are finite (N elements in C, $n = 1,2, \dots, N$)
- Probabilities of every outcome are objectively known
  - p_1 for outcome 1, p_2 for outcome 2, etc.

Simple Lotteries

- ***Simple lottery*** is a list
$$L = (p_1, p_2, \dots, p_N)$$
with $p_n \geq 0$ for all n and $\sum_{n=1}^{N} p_n = 1$, where p_n is interpreted as the probability of outcome n occuring.
- In some books, lotteries are described including the outcomes too.

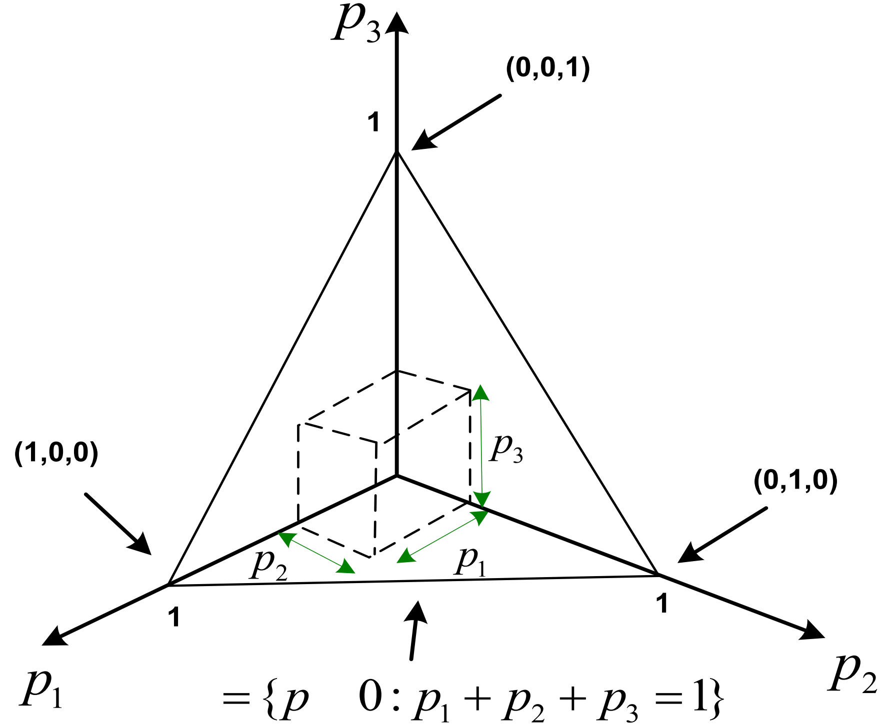
Simple Lotteries

- A simple lottery with 2 possible outcomes
- "Degenerated" probability pairs
  - at (0,1), outcome 2 happens with certainty.
  - at (1,0), outcome 1 happens with certainty.
- Strictly positive probability pairs
  - Individual faces some uncertainty, i.e., $p_1 + p_2 = 1$



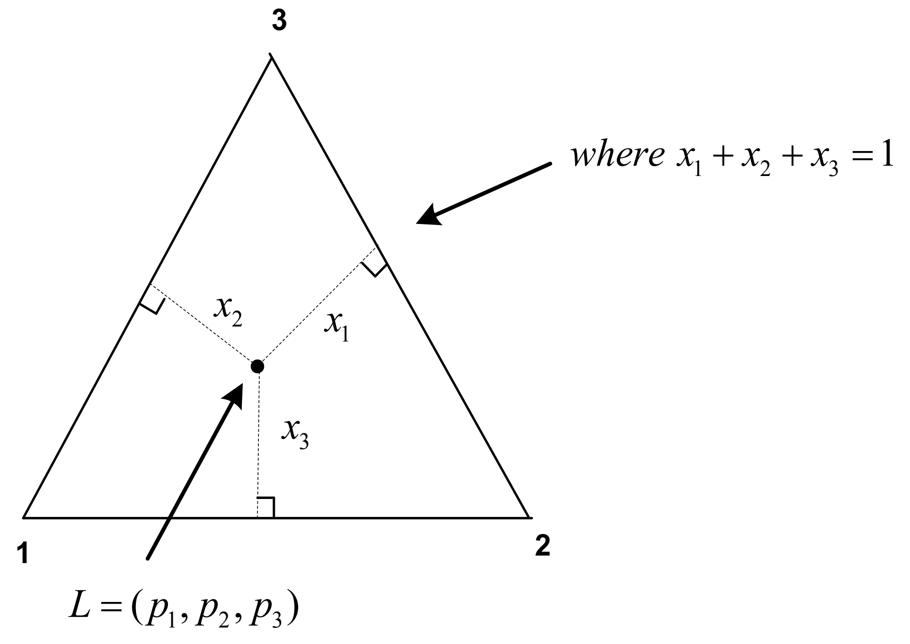
Simple Lotteries

- A simple lottery with 3 possible outcomes (i.e., 3-dim. simplex).
- Intercepts represent degenerated probabilities where one outcome is certain.
- Points strictly inside the hyperplane connecting the three intercepts denote a lottery where the individual faces uncertainty.



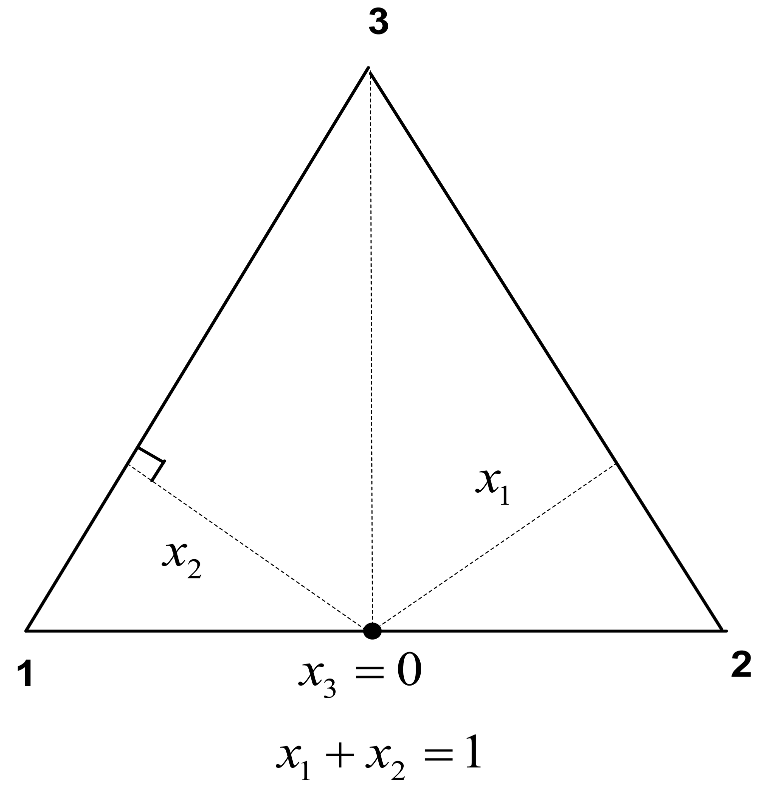
Simple Lotteries

- 2-dim. projection of the 3-dim. simplex
- Vertices represent the intercepts
- The distance from a given point to the side of the triangle measures the probability that the outcome represented at the opposite vertex occurs.



Simple Lotteries

- A lottery lies on one of the boundaries of the triangle:
  - We can only construct segments connecting the lottery to two of the outcomes.
  - The probability associated with the third outcome is zero.



Compound Lotteries

- Given simple lotteries
$$L_k = (p_1^k, p_2^k, \dots, p_N^k) \text{ for } k = 1,2,\dots,K$$
and probabilities $\alpha_k \geq 0$ with $\sum_{n=1}^{K} \alpha_k = 1$, then the ***compound lottery*** $(L_1, L_2, \dots, L_K; \alpha_1, \alpha_2, \dots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1,2,\dots,K$.
  - Think about a compound lottery as a "lottery of lotteries": first, I have probability α_k of playing lottery 1, and if that happens, I have probability p_1^k of outcome 1 occurring.
  - Then, the joint probability of outcome 1 is
$$p_1 = \alpha_1 \cdot p_1^1 + \alpha_2 \cdot p_1^2 + \dots + \alpha_K \cdot p_1^K$$

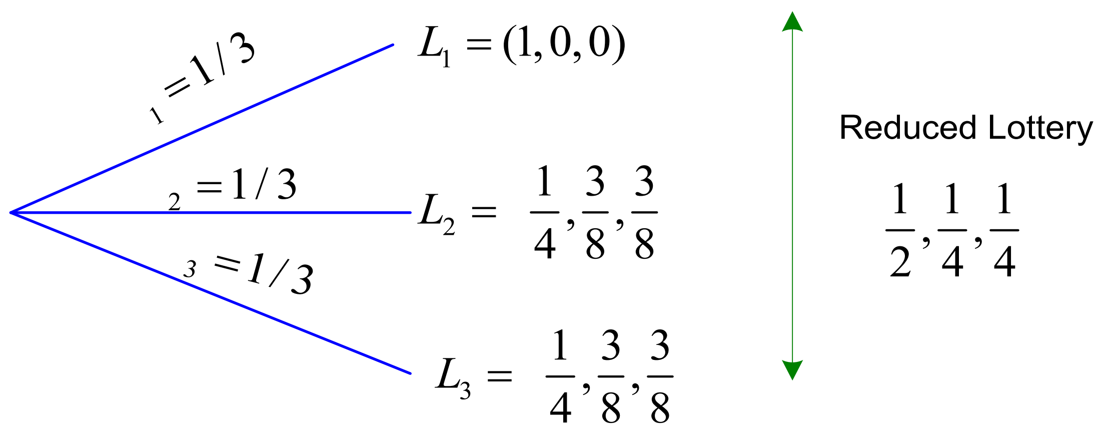
Compound and Reduced Lotteries

- Given that interpretation, the following result should come at no surprise:
  - For any compound lottery $(L_1, L_2, \dots, L_K; \alpha_1, \alpha_2, \dots, \alpha_K)$, we can calculate its corresponding ***reduced lottery*** as the simple lottery $L = (p_1, p_2, \dots, p_N)$ that generates the same ultimate probability distribution of outcomes.
- The reduced lottery L of any compound lottery can be obtained by

$$L = \alpha_1 L_1 + \alpha_2 L_2 + \cdots + \alpha_K L_K \in \Delta$$

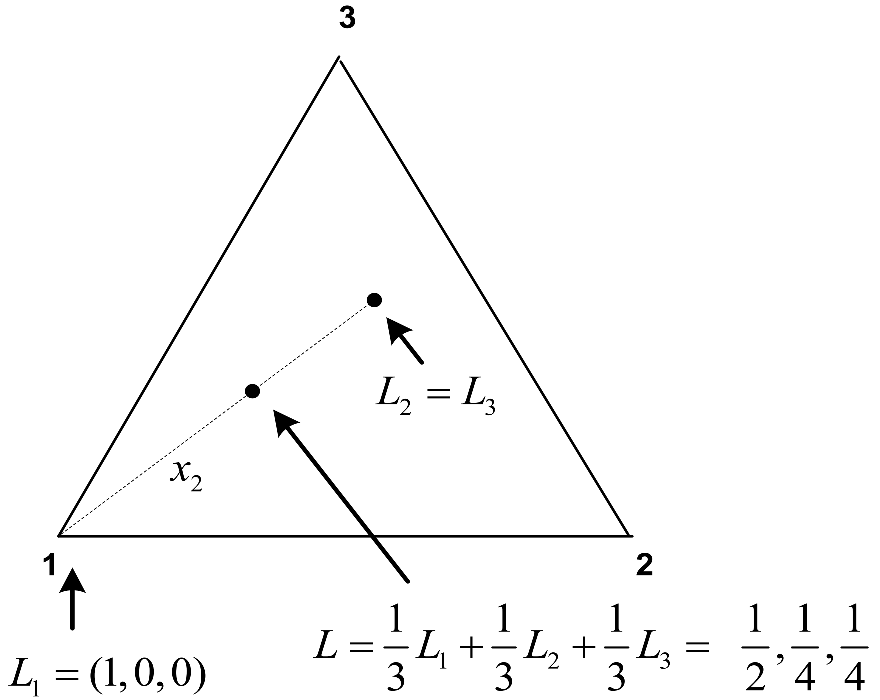
Compound and Reduced Lotteries

- ***Example 1***:
  - All three lotteries are equally likely
  - $P(\text{outcome } 1) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{2}$
  - $P(\text{outcome } 2) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{3}{8} + \frac{1}{3} \cdot \frac{3}{8} = \frac{1}{4}$
  - $P(\text{outcome } 3) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{3}{8} + \frac{1}{3} \cdot \frac{3}{8} = \frac{1}{4}$



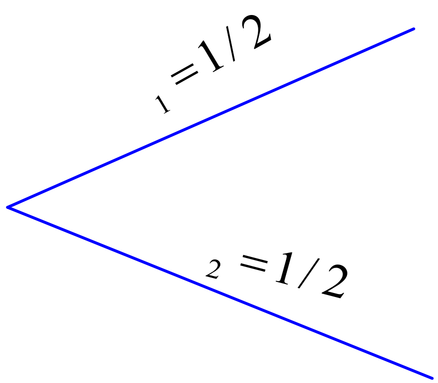
Compound and Reduced Lotteries

- ***Example 1*** (continued):
  - Probability simplex of the reduced lottery of a compound lottery
  - Reduced lottery L assigns the same probability weight to each simple lottery.



Compound and Reduced Lotteries

- ***Example 2***:
  - Both lotteries are equally likely



$$L_4 = \frac{1}{2}, \frac{1}{2}, 0$$

$$L_5 = \frac{1}{2}, 0, \frac{1}{2}$$

Reduced Lottery

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$$

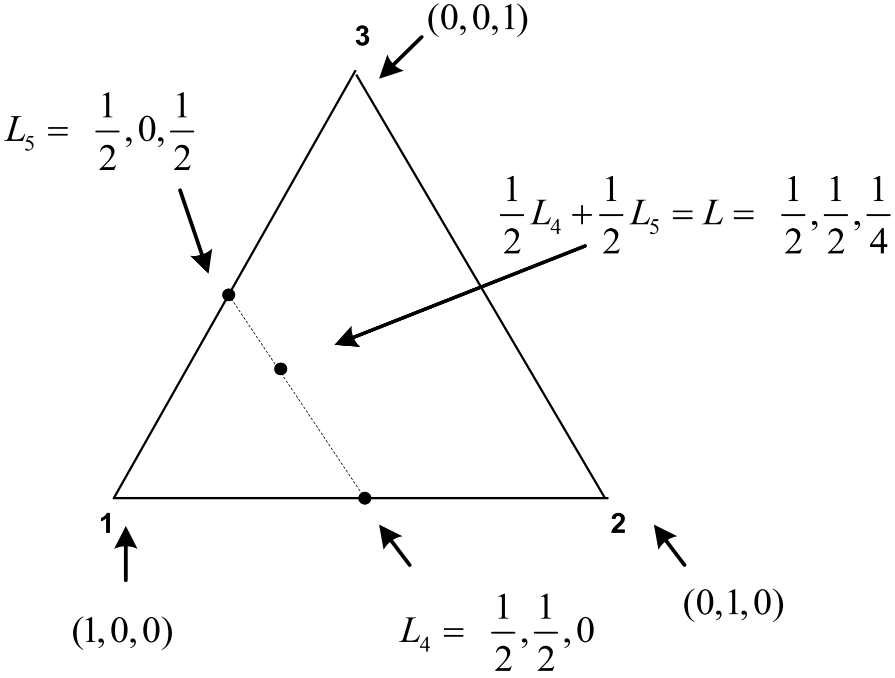
Outcome 1 $\longrightarrow \frac{1}{2} \; \frac{1}{2} + \frac{1}{2} \; \frac{1}{2} = \frac{1}{2}$

Outcome 2 $\longrightarrow \frac{1}{2} \; \frac{1}{2} + \frac{1}{2} \; 0 = \frac{1}{4}$

Outcome 3 $\longrightarrow \frac{1}{2} \; 0 + \frac{1}{2} \; \frac{1}{2} = \frac{1}{4}$

Compound and Reduced Lotteries

- ***Example 2*** (continued):
  - Probability simplex of the reduced lottery of a compound lottery



Compound and Reduced Lotteries

- Consumer is indifferent between the two compound lotteries which induce the same reduced lottery
  - This was illustrated in the previous Examples 1 and 2 where, despite facing different compound lotteries, the consumer obtained the same reduced lottery.
- We refer to this assumption as the ***Consequentialist hypothesis***:
  - Only consequences, and the probability associated to every consequence (outcome) matters, but not the route that we follow in order to obtain a given consequence.

Preferences over Lotteries

- For a given set of outcomes C, consider the set of all simple lotteries over C, $\mathcal{L}$.
- We assume that the decision maker has a *complete* and *transitive* preference relation $\succsim$ over lotteries in $\mathcal{L}$, allowing him to compare any pair of simple lotteries L and L'.
  - ***Completeness***: For any two lotteries L and L', either $L \succsim L'$ or $L' \succsim L$, or both.
  - ***Transitivity***: For any three lotteries L, L' and L'', if $L \succsim L'$ and $L' \succsim L''$, then $L \succsim L''$.

Preferences over Lotteries

- ***Extreme preference for certainty***:
  - $L \succsim L'$ if and only if

$$\max_{n \in N} p_n \geq \max_{n \in N} p'_n$$

  - The decision maker is only concerned about the probability associated with the most likely outcome.

Preferences over Lotteries

- ***Smallest size of the support***:
  - $L \gtrsim L'$ if and only if

$$\text{supp}(L) \leq \text{supp}(L')$$

where $\text{supp}(L) = \{n \in N\colon p_n > 0\}$.

  - The decision maker prefers the lottery whose probability distribution is concentrated over the smallest set of possible outcomes.

Preferences over Lotteries

- ***Lexicographic preferences***:
  - First, order outcomes from most preferred (outcome 1) to least preferred (outcome n).
  - Then $L \succsim L'$, if and only if

$$p_1 > p_1', \text{ or}$$
$$\text{If } p_1 = p_1' \text{ and } p_2 > p_2', \text{ or}$$
$$\text{If } p_1 = p_1' \text{ and } p_2 = p_2' \text{ and } p_3 > p_3', \text{ or}$$
$$\dots$$

  - The decision maker weakly prefers lottery L to L' if outcome 1 is more likely to occur in lottery L than in lottery L'.
  - If outcome 1 is as likely to occur in both lotteries, he moves to outcome 2, and so on.

Preferences over Lotteries

- ***The worst case scenario***:
  - First, attach a number $v(z)$ to every outcome $z \in C$, that is, $v(z) \in \mathbb{R}$.
  - Then $L \gtrsim L'$ if and only if

$$\min\{v(z) \colon p(z) > 0\} > \min\{v(z) \colon p'(z) > 0\}$$

  - The decision maker prefers lottery L if the lowest utility he can get from playing lottery L is higher than the lowest utility he can obtain from playing lottery L'.

Preferences over Lotteries

- Continuity of preferences over lotteries:
  - ***Continuity 1***: For any three lotteries L, L', and L'', the sets

    $\{\alpha \in [0,1]: \alpha L + (1-\alpha)L' \succsim L''\} \subset [0,1]$ and

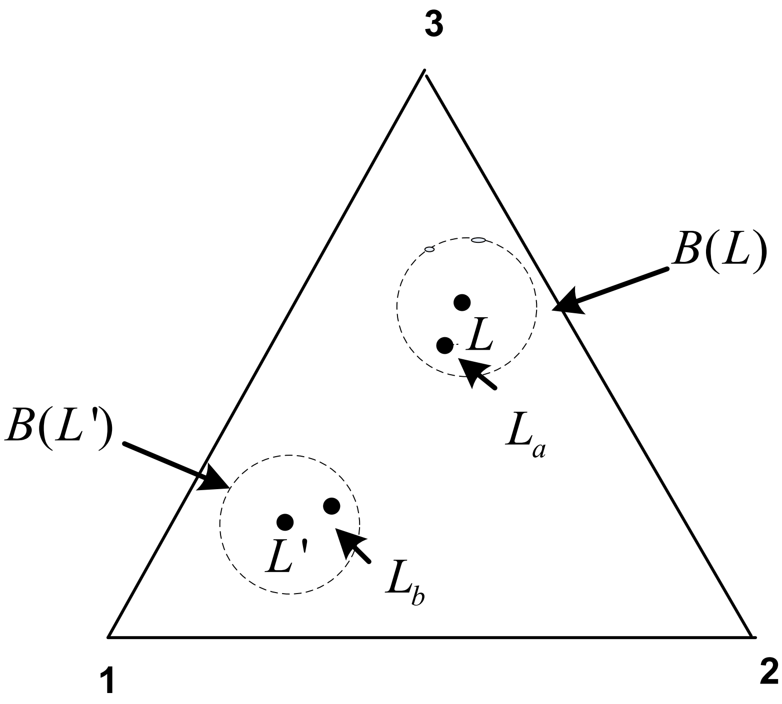
    $\{\alpha \in [0,1]: L'' \succsim \alpha L + (1-\alpha)L'\} \subset [0,1]$

    are closed.

  - ***Continuity 2***: if $L \succ L'$, then there is a neighborhoods of L and L', $B(L)$ and $B(L')$, such that for all $L_a \in B(L)$ and $L_b \in B(L')$, we have $L_a \succ L_b$.

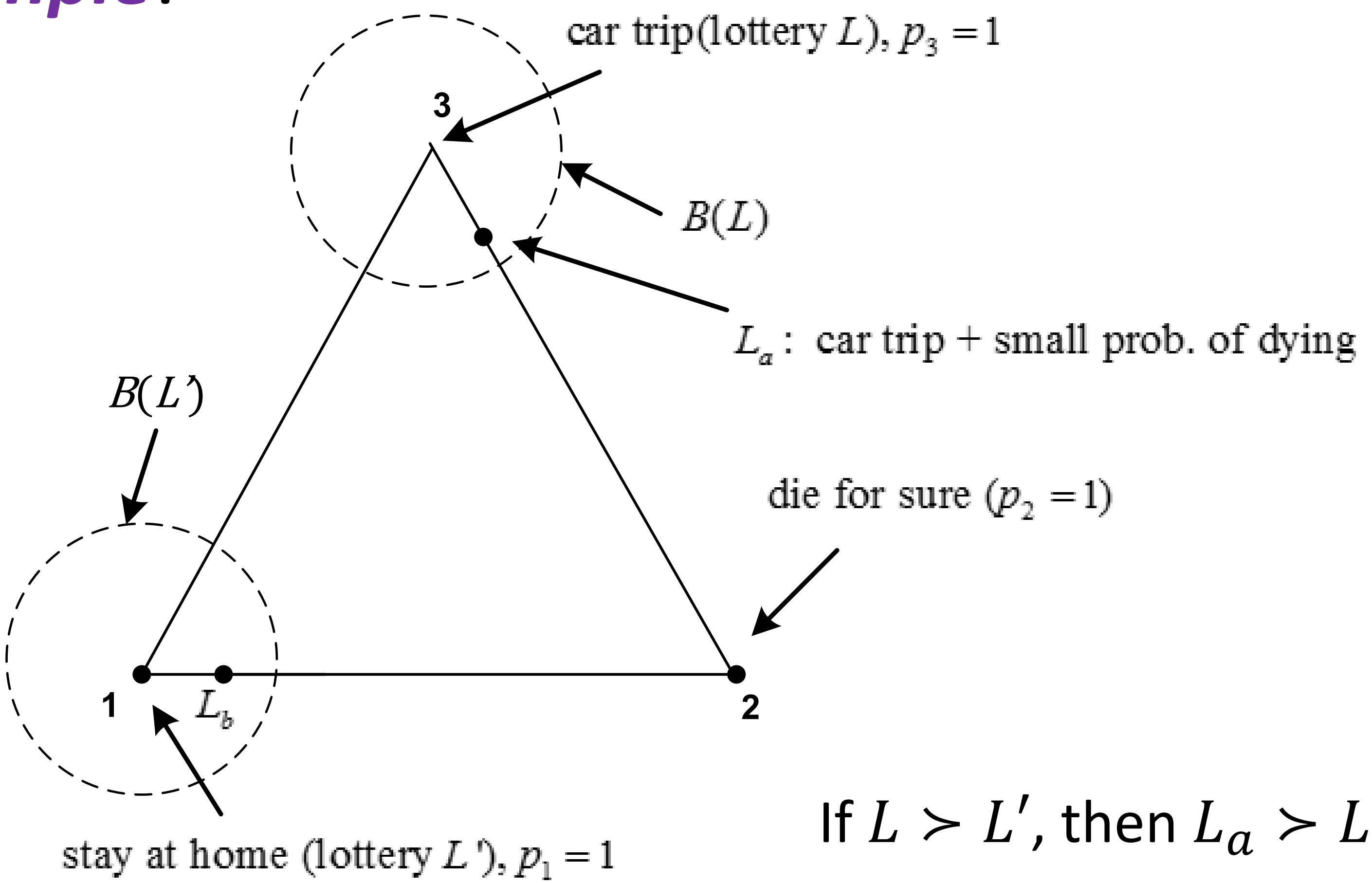
Preferences over Lotteries

- Small changes in the probability distribution of lotteries L and L' do not change the preference over the two lotteries.



Preferences over Lotteries

- ***Example***:



If $L \succ L'$, then $L_a \succ L_b$.

Preferences over Lotteries

- The continuity assumption, as in consumer theory, implies the existence of a utility function $U: \mathcal{L} \to \mathbb{R}$ such that

$$L \succsim L' \text{ if and only if } U(L) \geq U(L')$$

- However, we first impose an additional assumption in order to have a more structured utility function.
  - The following assumption is related with consequentialism: the *Independence axiom*.

Preferences over Lotteries

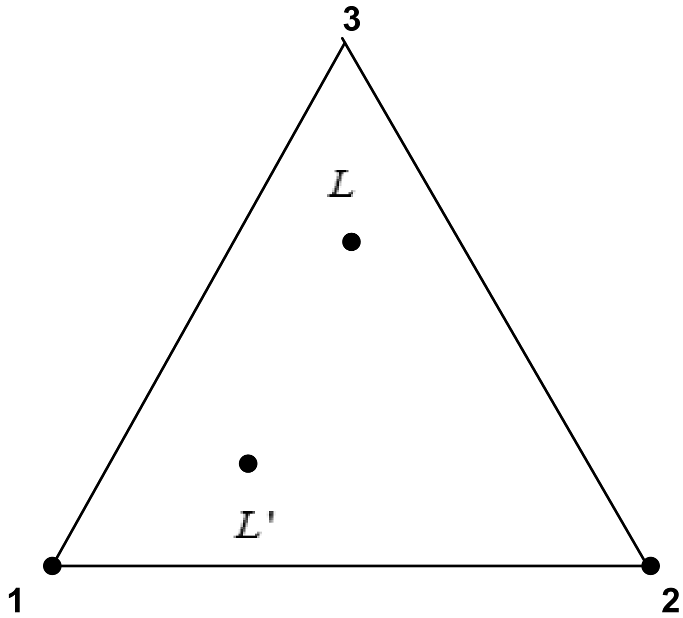
- ***Independence Axiom (IA)***: a preference relation satisfies IA if, for any three lotteries L, L', and L'', and $\alpha \in (0,1)$ we have

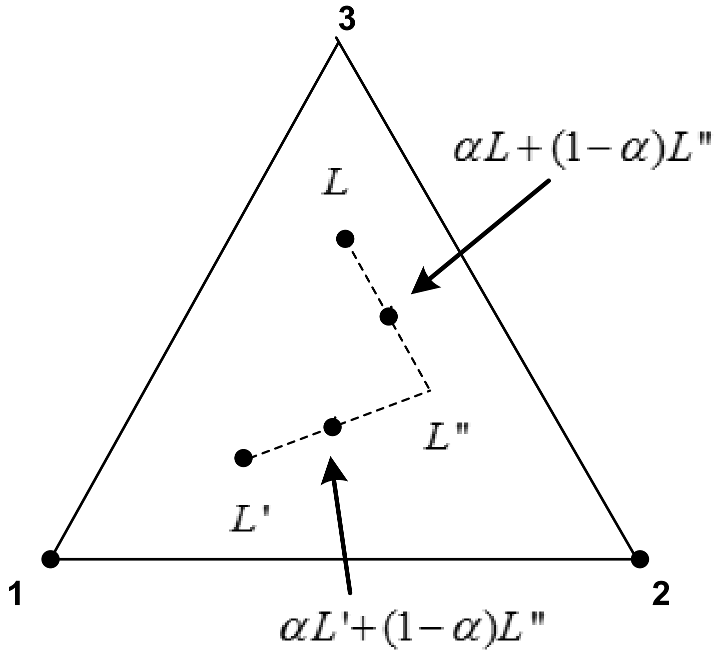
$$L \succsim L' \text{ if and only if}$$
$$\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

- *Intuition*: If we mix each of two lotteries, L and L', with a third one (L''), then the preference ordering of the two resulting compound lotteries is independent of the particular third lottery .

Preferences over Lotteries

- $L \succsim L'$ if and only if
  $$\alpha L + (1-\alpha)L' \succsim \alpha L' + (1-\alpha)L''$$





Preferences over Lotteries

- ***Example 1*** (intuition):
  - The decision maker prefers lottery L to L', $L \succsim L'$
  - Construct a compound lottery by a coin toss:
    - play lottery L if heads comes up
    - play lottery L'' if tails comes up
  - By IA, if $L \succsim L'$, then

$$\frac{1}{2}L + \frac{1}{2}L'' \succsim \frac{1}{2}L' + \frac{1}{2}L''$$

Preferences over Lotteries

- ***Example 2*** (violations of IA):
  - Extreme preference for certainty
  - Consider two simple lotteries L and L' for which $L \sim L'$.
  - Construct two compound lotteries for which

$$\frac{1}{2}L + \frac{1}{2}L \nsim \frac{1}{2}L' + \frac{1}{2}L$$

  - If $L \sim L'$, then it must be that

$$\max\{p_1, p_2, \dots, p_n\} = \max\{p'_1, p'_2, \dots, p'_n\}$$

Preferences over Lotteries

- ***Example 2*** (violations of IA):
  - Compound lottery $\frac{1}{2}L + \frac{1}{2}L$ coincides with simple lottery L.
  - Hence, $\max\{p_1, p_2, \ldots, p_n\}$ is used to evaluate lottery L.
  - But compound lottery $\frac{1}{2}L' + \frac{1}{2}L$ is a reduced lottery with associated probabilities

$$\max\left\{\frac{1}{2}p_1' + \frac{1}{2}p_1, \ldots, \frac{1}{2}p_n' + \frac{1}{2}p_n\right\}$$

which might *differ* from $\max\{p_1', p_2', \ldots, p_n'\}$.

Preferences over Lotteries

- ***Example 2*** (violations of IA, a numerical example):
  - Consider two simple lotteries
    $$L = (0.4, 0.5, 0.1),\ L = (0.5, 0, 0.5)$$
  - Hence,
    $$\max\{0.4, 0.5, 0.1\} = 0.5 = \max\{0.5, 0, 0.5\}$$
    implying that $L \sim L'$.
  - However, the compound lottery $\frac{1}{2}L' + \frac{1}{2}L$ entails probabilities
    $$\left(\frac{0.4 + 0.5}{2}, \frac{0.5 + 0}{2}, \frac{0.1 + 0.5}{2}\right) = (0.45, 0.25, 0.3)$$
    implying that $\max\{0.45, 0.25, 0.3\} = 0.45$.

Preferences over Lotteries

- ***Example 2*** (violations of IA, a numerical example):
  - Therefore,

$$\max\{0.4, 0.5, 0.1\} = 0.5 > 0.45 = \max\{0.45, 0.25, 0.3\}$$

and thus $L = \frac{1}{2}L + \frac{1}{2}L \succ \frac{1}{2}L' + \frac{1}{2}L$.

  - This violates the IA, which requires

$$\frac{1}{2}L + \frac{1}{2}L \sim \frac{1}{2}L' + \frac{1}{2}L$$

Preferences over Lotteries

- ***Example 3*** (violations of IA, "worst case scenario"):
  - Consider $L \succ L'$.
  - Then, the compound lottery $\frac{1}{2}L + \frac{1}{2}L$ does *not* need to be preferred to $\frac{1}{2}L' + \frac{1}{2}L$.
  - *Example*:
    - Consider the simple lotteries $L = (1{,}3)$ and $L' = (10{,}0)$, with probabilities (p_1, p_2) and (p_1', p_2'), respectively.
    - This implies

$$\min\{v(z) : p(z) > 0\} = 1 \text{ for lottery } L$$
$$\min\{v(z) : p'(z) > 0\} = 0 \text{ for lottery } L'$$

    - Hence, $L \succ L'$.

Preferences over Lotteries

- ***Example 3*** (violations of IA, "worst case scenario"):
  - *Example* (continued):
    - However, the compound lottery $\frac{1}{2}L + \frac{1}{2}L'$ is $\left(\frac{11}{2}, \frac{3}{2}\right)$, whose worst possible outcome is $\frac{3}{2}$, which is preferred to that of $\frac{1}{2}L + \frac{1}{2}L$, which is 1.
    - Hence, despite $L \succ L'$ over simple lotteries,

$$L = \frac{1}{2}L + \frac{1}{2}L \prec \frac{1}{2}L + \frac{1}{2}L',$$

which violates the IA.

Expected Utility Theory

Expected Utility Theory

- The utility function $U\colon \mathcal{L} \rightarrow \mathbb{R}$ has the ***expected utility*** (EU) form if there is an assignment of numbers $(u_1, u_2, \ldots, u_N)$ to the N possible outcomes such that, for every simple lottery $L = (p_1, p_2, \ldots, p_N) \in \mathcal{L}$ we have

$$U(L) = p_1 u_1 + \cdots + p_N u_N$$

  - A utility function with the EU form is also referred to as a **von-Neumann-Morgenstern** (vNM) expected utility function.
  - Note that this function is *linear* in the probabilities.

Expected Utility Theory

- Hence, a utility function $U: \mathcal{L} \rightarrow \mathbb{R}$ has the expected utility form if and only if it is *linear* in the probabilities, i.e.,

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k \cdot U(L_k)$$

for any K lotteries $L_k \in \mathcal{L}$, $k = 1,2, \ldots, K$ and probabilities $(\alpha_1, \alpha_2, \ldots, \alpha_K) \geq 0$ and $\sum_{k=1}^{K} \alpha_k = 1$.

- *Intuition*: the utility of the expected value of the K lotteries, $U\left(\sum_{k=1}^{K} \alpha_k L_k\right)$, coincides with the expected utility of the K lotteries, $\sum_{k=1}^{K} \alpha_k U(L_k)$.

Expected Utility Theory

- Note that the utility of the expected value of playing the K lotteries is

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_n u_n \cdot \left(\sum_k \alpha_k\, p_n^k\right)$$

where $\sum_k \alpha_k\, p_n^k$ is the total joint probability of outcome n occurring.

Expected Utility Theory

- Note that the expected utility from playing the K lotteries is

$$\sum_{k=1}^{K} \alpha_k \cdot U(L_k) = \sum_{k} \alpha_k \cdot \left(\sum_{n} u_n \, p_n^k \right)$$

where $\sum_n u_n \, p_n^k$ is the expected utility from playing a given lottery k.

Expected Utility Theory

- The EU property is a *cardinal property*:
  - Not only rank matters, the particular number resulting form $U: \mathcal{L} \to \mathbb{R}$ also matters.
- Hence, the EU form is preserved only under increasing linear transformations (a.k.a. affine transformations).
  - Hence, the expected utility function $\tilde{U}: \mathcal{L} \to \mathbb{R}$ is another vNM utility function if and only if

$$\tilde{U}(L) = \beta U(L) + \gamma$$

for every $L \in \mathcal{L}$, where $\beta > 0$.

Expected Utility Theory: Representability

- Suppose that the preference relation $\succsim$ satisfies rationality, continuity and independence. Then, $\succsim$ admits a utility representation of the EU form.
- That is, we can assign a number u_n to every outcome $n = 1,2, \dots, N$ in such a manner that for any two lotteries

  $$L = (p_1, p_2, \dots, p_N) \text{ and } L' = (p_1', p_2' \dots, p_N')$$

  we have $L \succsim L'$ if and only if $U(L) \geq U(L')$, or

  $$\sum_{n=1}^{N} p_n u_n \geq \sum_{n=1}^{N} p_n' u_n$$

- *Notation*: u_n is the utility that the decision maker assigns to outcome n. It is usually referred as the Bernoulli utility function.

Expected Utility Theory: Indifference Curves

- Let us next analyze the effect of the IA on indifference curves over lotteries.

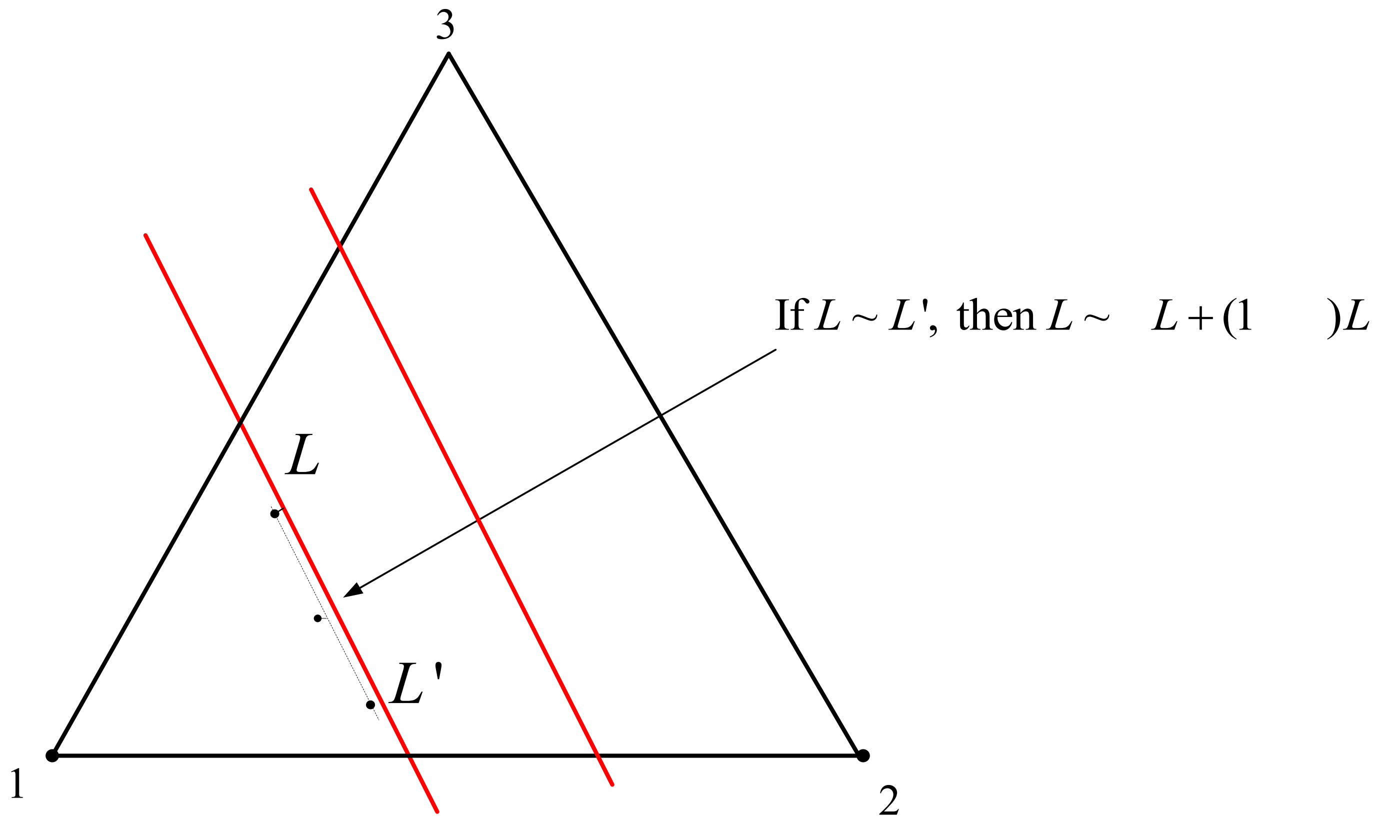
1) ***Indifference curves must be straight lines***:

   Recall that from the IA, $L \sim L'$ implies that

$$\underbrace{\alpha L + (1-\alpha)L}_{L} \sim \alpha L + (1-\alpha)L'$$

   for all $\alpha \in (0,1)$.

Expected Utility Theory: Indifference Curves



Straight indifference curves

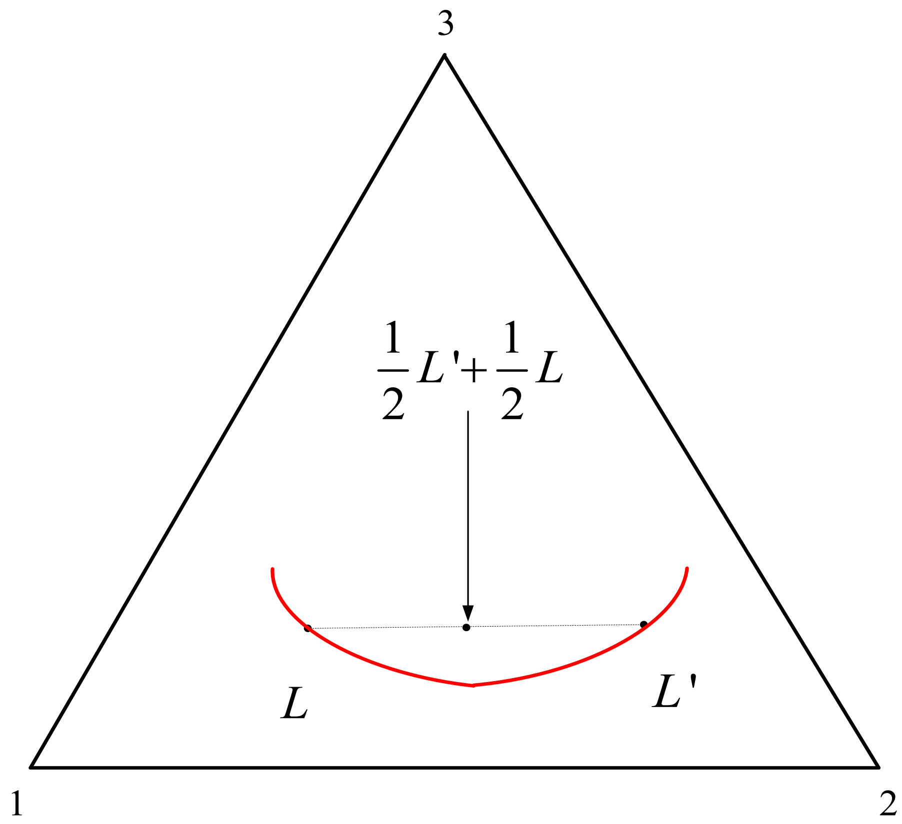
Expected Utility Theory: Indifference Curves

- Why indifference curves must be straight?
  - We have that $L \sim L'$, but $L \prec \frac{1}{2}L + \frac{1}{2}L'$. This is equivalent to
$$\frac{1}{2}L + \frac{1}{2}L \prec \frac{1}{2}L + \frac{1}{2}L'$$
  - But from the IA we must have
$$\frac{1}{2}L + \frac{1}{2}L \sim \frac{1}{2}L + \frac{1}{2}L'$$
  - Hence, indifference curves must be straight lines in order to satisfy the IA.

Expected Utility Theory: Indifference Curves

- Curvy indifference curves over lotteries are incompatible with the IA
  - The compound lottery $\frac{1}{2}L + \frac{1}{2}L'$ would not lie on the same indifference curve as lottery L and L'.
  - Hence, the decision maker is *not* indifferent between the compound lotteries $\frac{1}{2}L + \frac{1}{2}L'$ and $\frac{1}{2}L + \frac{1}{2}L'$.

Expected Utility Theory: Indifference Curves



Curvy indifference curve

Expected Utility Theory: Indifference Curves

2) *Indifference curves must be parallel lines*:

If we have that $L \sim L'$, then by the IA

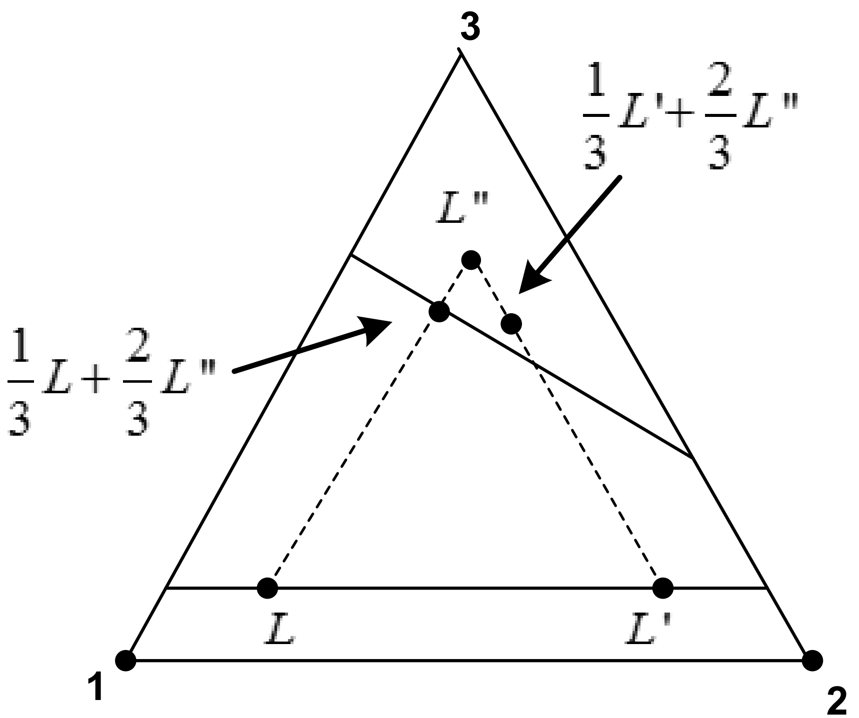
$$\frac{1}{3}L + \frac{2}{3}L'' \sim \frac{1}{3}L' + \frac{2}{3}L''$$

- That is, the convex combination of L and L' with a third lottery L'' should also lie on the same indifference curve.

- This implies that the indifference curves must be parallel lines in order to satisfy the IA.

Expected Utility Theory: Indifference Curves

- Nonparallel indifference curves are incompatible with the IA.
  - If compound lotteries $\frac{1}{3}L + \frac{2}{3}L''$ and $\frac{1}{3}L' + \frac{2}{3}L''$ lie on different (nonparallel) indifference curves, then
  $$\frac{1}{3}L + \frac{2}{3}L'' \prec \frac{1}{3}L' + \frac{2}{3}L''$$
  which violates the IA.



Expected Utility Theory:

Violations of the IA:

- Despite the intuitive appeal of the IA, we encounter several settings in which decision makers violate it.
- We next elaborate on these violations.

Expected Utility Theory: Violations of the IA

- ***Allais' paradox***:
  - Consider a lottery over three possible monetary outcomes:

1st prize	2nd prize	3rd prize
$2.5mln	**$500,000**	**$0**

  - First choice set:

$$L_1 = (0,1,0) \text{ and } L_1' = \left(\frac{10}{100}, \frac{89}{100}, \frac{1}{100}\right)$$

  - Second choice set:

$$L_2 = \left(0, \frac{11}{100}, \frac{89}{100}\right) \text{ and } L_2' = \left(\frac{10}{100}, 0, \frac{90}{100}\right)$$

Expected Utility Theory: Violations of the IA

- About 50% students surveyed expressed $L_1 \succ L_1'$ and $L_2' \succ L_2$.
- These choices violate the IA.
- To see this, consider that the decision maker's preferences over lotteries have a EU form. Hence, $L_1 \succ L_1'$ implies

$$u_5 > \frac{10}{100}u_{25} + \frac{89}{100}u_5 + \frac{1}{100}u_0$$

- By the IA, we can add $\frac{89}{100}u_0 - \frac{89}{100}u_5$ on both sides

$$u_5 + \left(\frac{89}{100}u_0 - \frac{89}{100}u_5\right) >$$
$$\frac{10}{100}u_{25} + \frac{89}{100}u_5 + \frac{1}{100}u_0 + \left(\frac{89}{100}u_0 - \frac{89}{100}u_5\right)$$

Expected Utility Theory: Violations of the IA

– Simplifying

$$\underbrace{\frac{11}{100}u_5 + \frac{89}{100}u_0}_{\text{EU of } L_2} > \underbrace{\frac{10}{100}u_{25} + \frac{90}{100}u_0}_{\text{EU of } L_2'}$$

which implies $L_2 \succ L_2'$.

– Did your own choices violate the IA?

Expected Utility Theory: Violations of the IA

- Reactions to the Allais' Paradox:
  - *Approximation to rationality*: people adapt their choices as they go.
  - *Little economic significance*: the lotteries involve probabilities that are close to zero and one.
  - *Regret theory*: the reason why $L_1 \succ L_1'$ is because I didn't want to regret a sure win of $500,000.
  - Give up the IA in favor of a weaker assumption: the *betweenness axiom*.

Expected Utility Theory: Violations of the IA

- ***Machina's paradox***:
  - Consider that

    Trip to Barcelona $\succ$ Movie about Barcelona $\succ$ Home

  - Now, consider the following two lotteries

    $$L_1 = (\frac{99}{100}, \frac{1}{100}, 0) \text{ and } L_2 = (\frac{99}{100}, 0, \frac{1}{100})$$

  - From the previous preferences over certain outcomes, how can we know this individual's preferences over lotteries?
    - Using the IA.

Expected Utility Theory: Violations of the IA

– From $T \succ M$ and the IA, we can construct the compound lotteries

$$\frac{99}{100}T + \frac{1}{100}M \succ \frac{99}{100}M + \frac{1}{100}M$$

– From $M \succ H$ and the IA, we have

$$\frac{99}{100}M + \frac{1}{100}M \succ \frac{99}{100}M + \frac{1}{100}H$$

– By transitivity,

$$\underbrace{\frac{99}{100}T + \frac{1}{100}M}_{L_1} \succ \underbrace{\frac{99}{100}T + \frac{1}{100}H}_{L_2}$$

– Hence, $L_1 \succ L_2$.

Expected Utility Theory: Violations of the IA

- Therefore, for preferences over lotteries to be consistent with the IA, we need $L_1 \succ L_2$.
- Many subjects in experimental settings would rather prefer L_2, thus violating the IA.
- Many people explain choosing L_2 over L_1 on grounds of the disappointment they would experience in the case of losing the trip to Barcelona, and having to watch a movie instead.
  - Similar to regret theory.

Expected Utility Theory: Violations of the IA

- ***Dutch books***:
  - In the above two anomalies, actual behavior is inconsistent with the IA.
  - Can we then rely on the IA?
  - What would happen to individuals whose behavior violates the IA?
  - They would be weeded out of the market because they would be open to the acceptance of so-called *Dutch books*, leading them to a sure loss of money.

Expected Utility Theory: Violations of the IA

- Consider that $L \succ L'$. By the IA, we should have
$$\underbrace{\alpha L + (1-\alpha)L}_{L} \succ \alpha L + (1-\alpha)L'$$
- If, instead, the IA is violated, then
$$L \prec \alpha L + (1-\alpha)L'$$
- Consider an individual with these preferences, who initially owns lottery L.
- If we offer him the compound lottery $\alpha L + (1-\alpha)L'$, for a small fee \$$x$, he would accept such a trade.

Expected Utility Theory: Violations of the IA

– After the realization stage, he owns either L or L'
  - If L', then we offer L again for \$$y$.
  - If L, then we offer $\alpha L + (1-\alpha)L'$ for \$$y$.
– Either way, he is at the same position as he started (owning L or $\alpha L + (1-\alpha)L'$), but having lost \$$x$ + \$$y$ in the process.
– We can repeat this process ad infinitum.
– Hence, individuals with preferences that violate the IA would be exploited by microeconomists (they would be a "money pump").

Expected Utility Theory: Violations of the IA

- Further reading:
  - "Developments in non-expected utility theory: The hunt for a descriptive theory of choice under risk" (2000) by Chris Starmer, *Journal of Economic Literature*, vol. 38(2)
  - *Choices, Values and Frames* (2000) by Nobel prize winners Daniel Kahneman and Amos Tversky, Cambridge University Press.
  - *Theory of Decision under Uncertainty* (2009) by Itzhak Gilboa, Cambridge University Press.

Theories Modifying Expected Utility Theory

1) Weighted utility theory:

- The payoff function from playing lottery L is

$$V(L) = \sum_{x \in C} w_i \cdot u(x_i)$$

where

$$w_i = \frac{g(x_i)p(x_i)}{\sum_{x \in C} g(x_i)p(x_i)} \text{ and } g: C \to \mathbb{R}$$

- The utility of outcome $x_i \in C$ is weighted according to:
  a) its probability $p(x_i)$
  b) outcome x_i itself through function $g: C \to \mathbb{R}$

Theories Modifying Expected Utility Theory

– ***Example***: Consider a lottery with two payoffs x_1 and x_2 with probabilities p and $1-p$. Then, the weighted utility is

$$V(L) = w_1 u(x_1) + w_2 u(x_2)$$
$$= \frac{g(x_1)p}{g(x_1)p + g(x_2)(1-p)} u(x_1)$$
$$+ \frac{g(x_2)(1-p)}{g(x_1)p + g(x_2)(1-p)} u(x_2)$$

If $g(x_i) = g(x_j)$ for any $x_i \neq x_j$, then

$$V(L) = pu(x_1) + (1-p)u(x_2)$$

which is a standard expected utility function.

Theories Modifying Expected Utility Theory

- The *weighted utility theory* relies on the same axioms as *expected utility theory*, except for the IA, which is relaxed to the "weak independence axiom."
  - ***Weak independence axiom***: if we have that $L_1 \sim L_2$, we can find a pair of probabilities α and α' such that

    $$\alpha L_1 + (1-\alpha)L_3 \sim \alpha' L_2 + (1-\alpha')L_3$$

  - The IA becomes a special case if $\alpha = \alpha'$.

Theories Modifying Expected Utility Theory

2) Rank dependent utility theory:

- First, rank the outcomes $x_1, x_2, \dots, x_n$ from worst (x_1) to best (x_n)
- Second, apply a probability weighting function

$$w_i = \pi(p_i + \cdots + p_n) - \pi(p_{i+1} + \cdots + p_n)$$
$$w_n = \pi(p_n)$$

  where $\pi(\cdot)$ is a non-decreasing *transformation function*, with $\pi(0) = 0$ and $\pi(1) = 1$.
- Finally, a rank-dependent utility is

$$V(L) = \sum_{x \in C} w_i \cdot u(x_i)$$

Theories Modifying Expected Utility Theory

– For a lottery with two outcomes, x_1 and x_2 where $x_2 > x_1$, the rank-dependent utility is

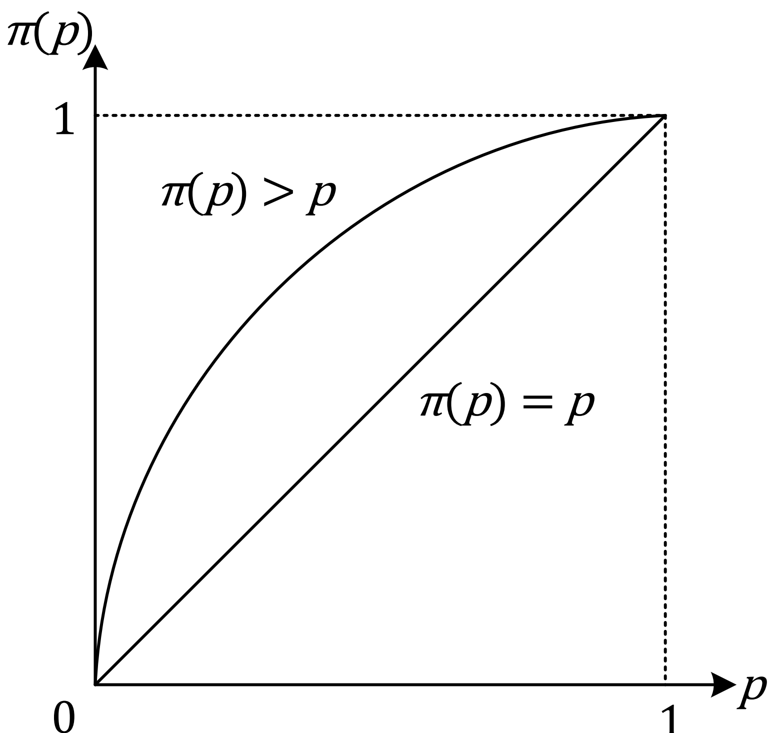
$$V(L) = w(p)u(x_1) + (1 - w(p))u(x_2)$$

where p is the probability of outcome x_1.

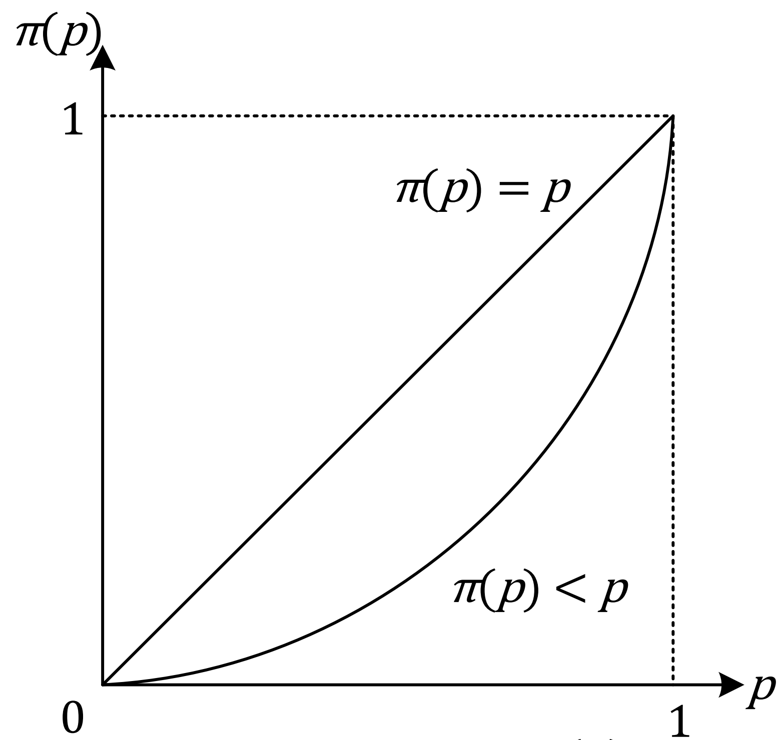
– This model allows for *different weight* to be attached to each outcome, as opposed to expected utility theory models in which the *same utility weight* is attached to all outcomes.

Theories Modifying Expected Utility Theory

– Transformation function $\pi(\cdot)$



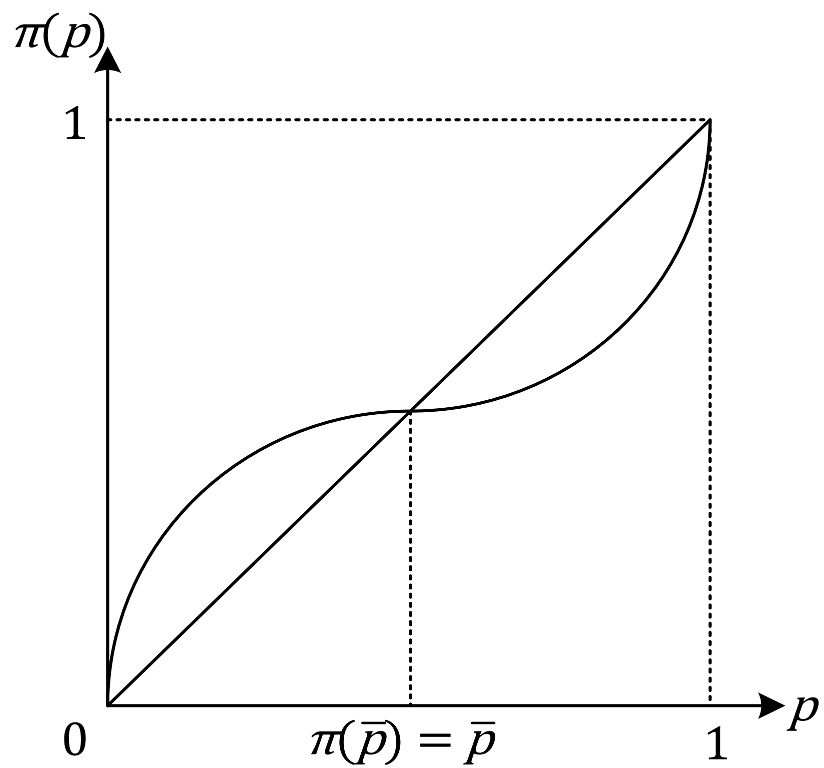
Optimistic $\pi(p)$



Pessimistic $\pi(p)$

Theories Modifying Expected Utility Theory

- Empirical evidence suggests an S-shaped transformation function.
- *Intuition*: individuals are pessimistic in rare outcomes (i.e., $p < \overline{p}$), but become optimistic for outcomes they have frequently encountered.



Theories Modifying Expected Utility Theory

- The *rank-dependent utility theory* relies on the same axioms as *expected utility theory*, except for the IA, which is replaced by co-monotonic independence.

Money Lotteries

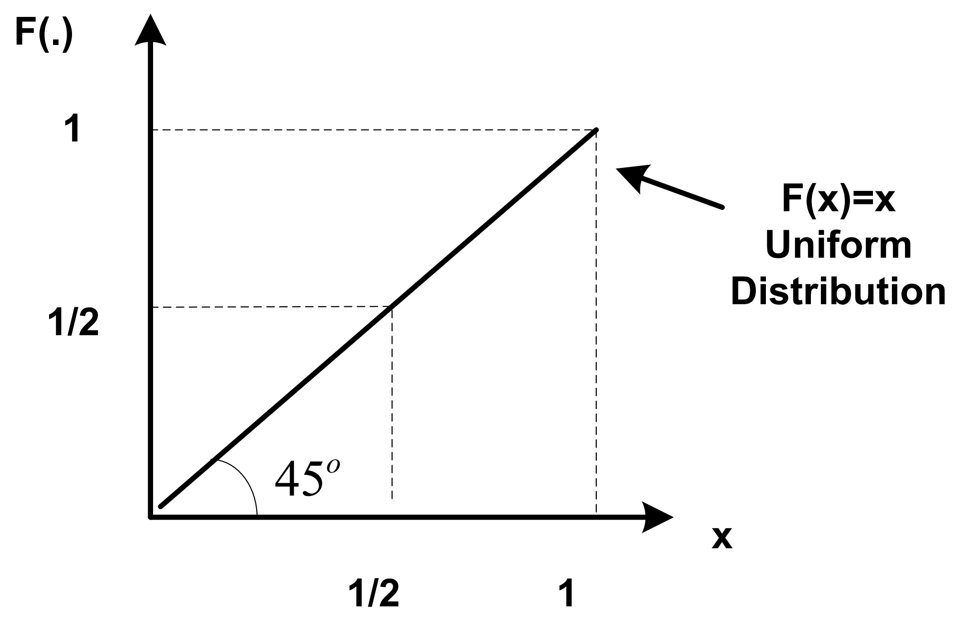
Money Lotteries

- We now restrict our attention to lotteries over *monetary* amounts, i.e., $C = \mathbb{R}$.
- Money is continuous variable, $x \in \mathbb{R}$, with cumulative distribution function (CDF)

$$F(x) = Prob\{y \leq x\} \text{ for all } y \in \mathbb{R}$$

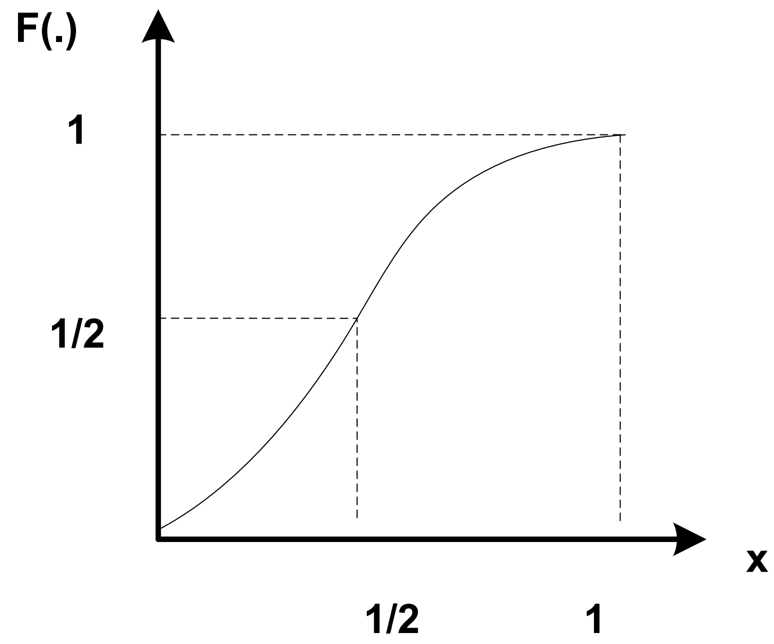
Money Lotteries

- A uniform, continuous CDF, $F(x) = x$
  - Same probability weight to every possible payoff



Money Lotteries

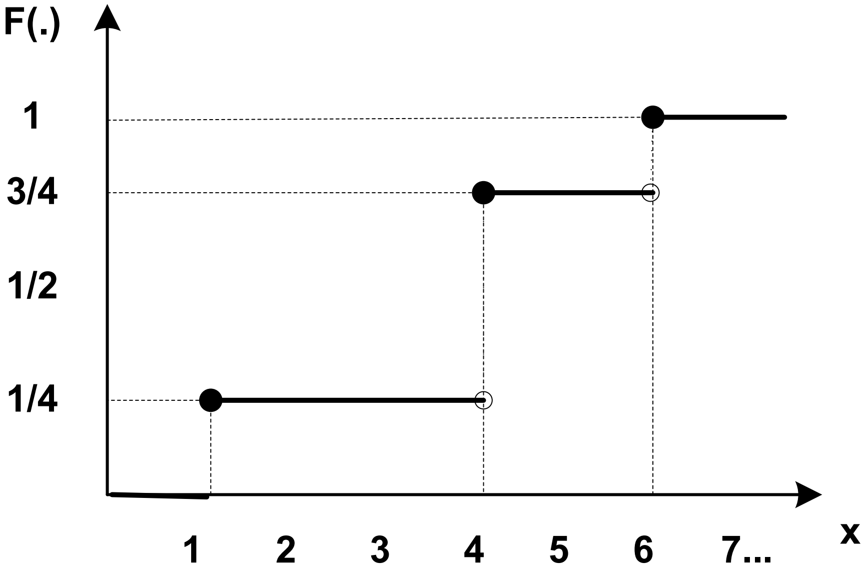
- A non-uniform, continuous CDF, $F(x)$



Money Lotteries

- A non-uniform, discrete CDF

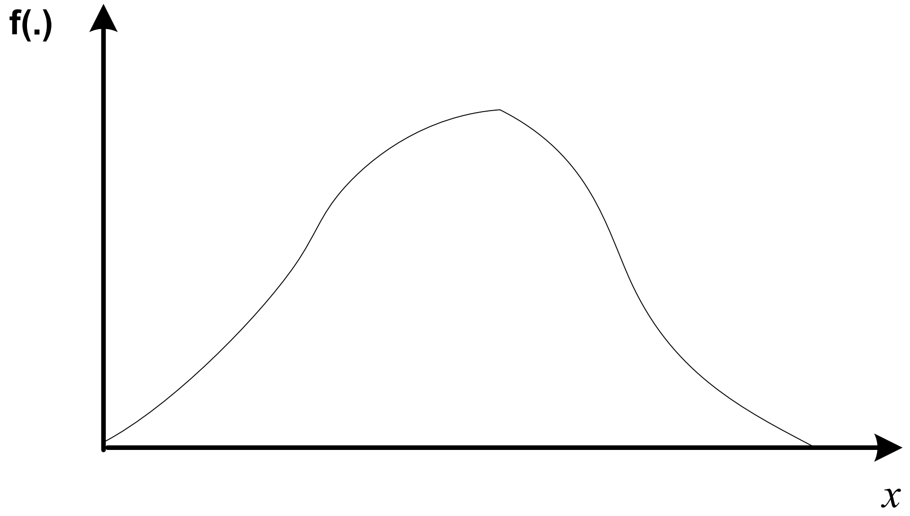
$$F(x) = \begin{cases} 0 \text{ if } x < 1 \\ \dfrac{1}{4} \text{ if } x \in [1, 4) \\ \dfrac{3}{4} \text{ if } x \in [4, 6) \\ 1 \text{ if } x \geq 6 \end{cases}$$



Money Lotteries

- If $f(x)$ is a density function associated with the *continuous* CDF $F(x)$, then

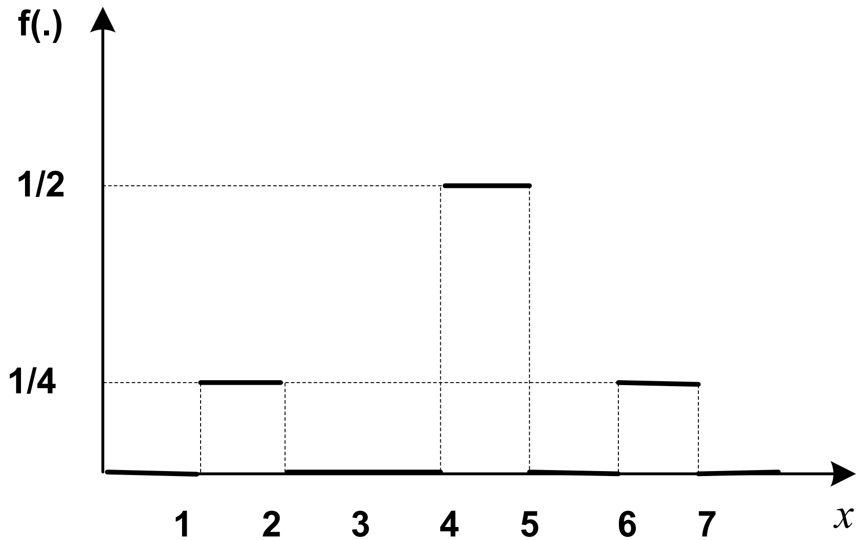
$$F(x) = \int_{-\infty}^{x} f(t)dt$$



Money Lotteries

- If $f(x)$ is a density function associated with the *discrete* CDF $F(x)$, then

$$F(x) = \sum_{t<x} f(t)$$



Money Lotteries

- We can represent *simple lotteries* by $F(x)$.
- *For compound lotteries*:
  - If the list of CDF's $F_1(x)$, $F_2(x)$, …, $F_K(x)$ represent K simple lotteries, each occurring with probability $\alpha_1, \alpha_2, \dots, \alpha_K$, then the compound lottery can be represented as

$$F(x) = \sum_{k=1}^{K} \alpha_k F_k(x)$$

  - For simplicity, assume that CDF's are distributed over non-negative amounts of money.

Money Lotteries

– We can express EU as

$$EU(F) = \int u(x)f(x)dx \text{ or } \int u(x)dF(x)$$

where $u(x)$ is an assignment of utility value to every non-negative amount of money.

– If there is a density function $f(x)$ associated with the CDF $F(x)$, then we can use either of the expressions. If there is no, we can only use the latter.

– *Note*: we do not need to write down the limits of integration, since the integral is over the full range of possible realizations of x.

Money Lotteries

- $EU(F)$ is the mathematical expectation of the values of $u(x)$, over all possible values of x.
- $EU(F)$ is linear in the probabilities
  - In the discrete probability distribution,
$$EU(F) = p_1(u_1) + p_2(u_2) + \cdots$$
- The EU representation is sensitive not only to the *mean* of the distribution, but also to the *variance*, and *higher order moments* of the distribution of monetary payoffs.
  - Let us next analyze this property.

Money Lotteries

- ***Example***: Let us show that if $u(x) = \beta x^2 + \gamma x$, then EU is determined by the mean and the variance alone.
  - Indeed,

$$EU(x) = \int u(x)dF(x) = \int [\beta x^2 + \gamma x]dF(x)$$

$$= \beta \underbrace{\int x^2 dF(x)}_{E(x^2)} + \gamma \underbrace{\int x\, dF(x)}_{E(x)}$$

  - On the other hand, we know that

$$Var(x) = E(x^2) - \left(E(x)\right)^2 \Longrightarrow$$
$$E(x^2) = Var(x) + \left(E(x)\right)^2$$

Money Lotteries

- ***Example*** (continued):
  - Substituting $E(x^2)$ in $EU(x)$,

$$EU(x) = \underbrace{\beta Var(x) + \beta\left(E(x)\right)^2}_{\beta E(x^2)} + \gamma E(x)$$

  - Hence, the EU is determined by the mean and the variance alone.

Money Lotteries

- Recall that we refer to $u(x)$ as the Bernoulli utility function, while $EU(x)$ is the vNM function.
- We imposed few assumptions on $u(x)$:
  - Increasing in money and continuous
- We must impose an additional assumption:
  - $u(x)$ is bounded
  - Otherwise, we can end up in relatively absurd situations (*St. Petersburg-Menger paradox*).

Money Lotteries

- ***St. Petersburg-Menger paradox***:
  - Consider an unbounded Bernoulli utility function, $u(x)$. Then, we can always find an amount of money x_m such that $u(x_m) > 2^m$, for every integer m.
  - Now consider a lottery in which we toss a coin repeatedly until tails come up. We give a monetary payoff of x_m if tails is obtained at the mth toss.
  - The probability that tails comes up in the m-th toss is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \ldots (m \text{ times}) = \frac{1}{2^m}$.

Money Lotteries

– Then, the EU of this lottery is

$$EU(x) = \sum_{m=1}^{\infty} \frac{1}{2^m} u(x_m)$$

– But, because of $u(x_m) > 2^m$, we have that

$$EU(x) = \sum_{m=1}^{\infty} \frac{1}{2^m} u(x_m) \geq \sum_{m=1}^{\infty} \frac{1}{2^m} 2^m$$

$$= \sum_{m=1}^{\infty} 1 = +\infty$$

which implies that this individual would be willing to pay infinite amounts of money to be able to pay this lottery.

– Hence, we assume that the Bernouilli utility function is bounded.

Measuring Risk Preferences

Measuring Risk Preferences

- An individual exhibits ***risk aversion*** if

$$\int u(x)dF(x) \leq u\left(\int xdF(x)\right)$$

  for any lottery $F(\cdot)$

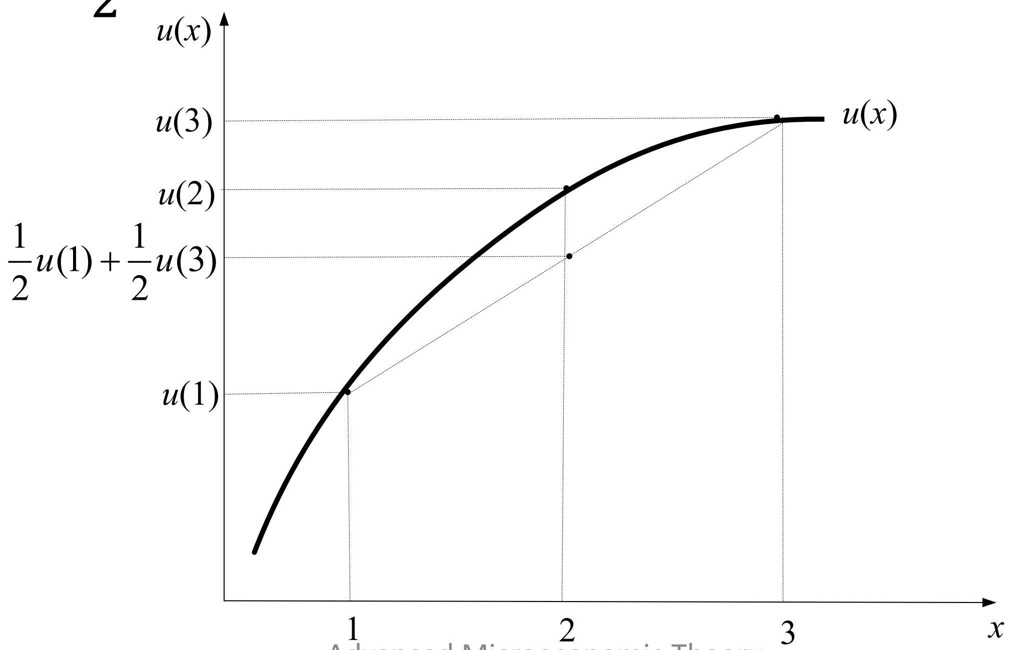
- *Intuition*:
  - The utility of receiving the expected monetary value of playing the lottery (left-hand side) is higher than…
  - The expected utility from playing the lottery (right-hand side).
- If this relationship happens with
  $a)$ $=$, we denote this individual as **risk neutral**
  $b)$ $<$, we denote him as **risk averter**
  $c)$ $\geq$, we denote him as **risk lover**.

Measuring Risk Preferences

- Graphical illustration:
  - Consider a lottery with two equally likely outcomes, \$1 and \$3, with associated utilities of $u(1)$ and $u(3)$, respectively.
  - *Expected value of the lottery* is $EV = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2$, with associated utility of $u(2)$.
  - *Expected utility of the lottery* is $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.

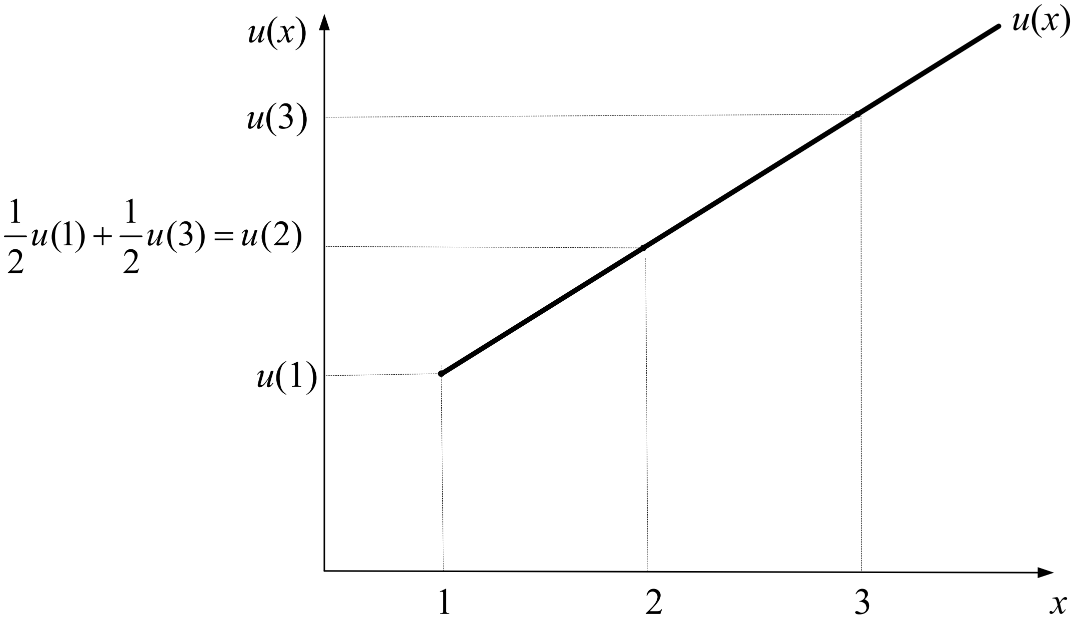
Measuring Risk Preferences

- *Risk averse individual*
  - Utility from the expected value of the lottery, $u(2)$, is **higher** than the EU from playing the lottery, $\frac{1}{2}u(1)+\frac{1}{2}u(3)$.



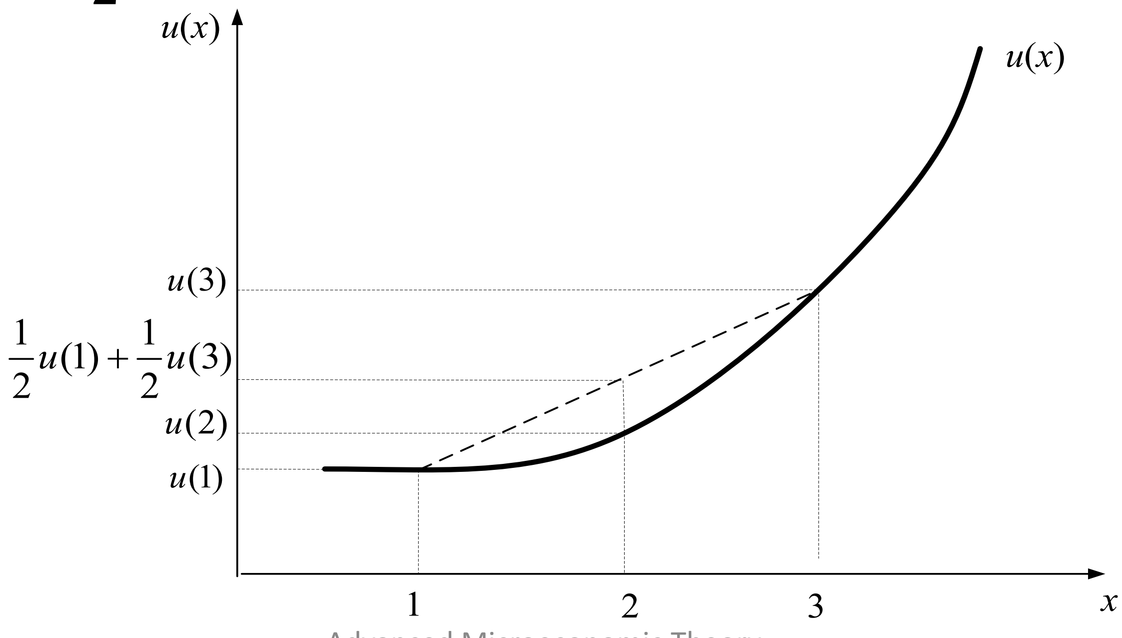
Measuring Risk Preferences

- *Risk neutral individual*
  - Utility from the expected value of the lottery, $u(2)$, **coincides** with the EU of playing the lottery, $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.



Measuring Risk Preferences

- *Risk loving individual*
  - Utility from the expected value of the lottery, $u(2)$, is **lower** than the EU from playing the lottery, $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.



Measuring Risk Preferences

- ***Certainty equivalent***, $c(F, u)$:
  - An alternative measure of risk aversion
  - It is the amount of money that makes the individual indifferent between playing the lottery $F(\cdot)$, and accepting a certain amount $c(F, u)$. That is,

$$u(c(F,u)) = \int u(x)dF(x) \text{ or } \sum u(x)f(x)$$

  - $c(F, u)$ is below (above) the expected value of the lottery for risk averse (lover) individuals, and exactly coincides for risk neutral individuals.

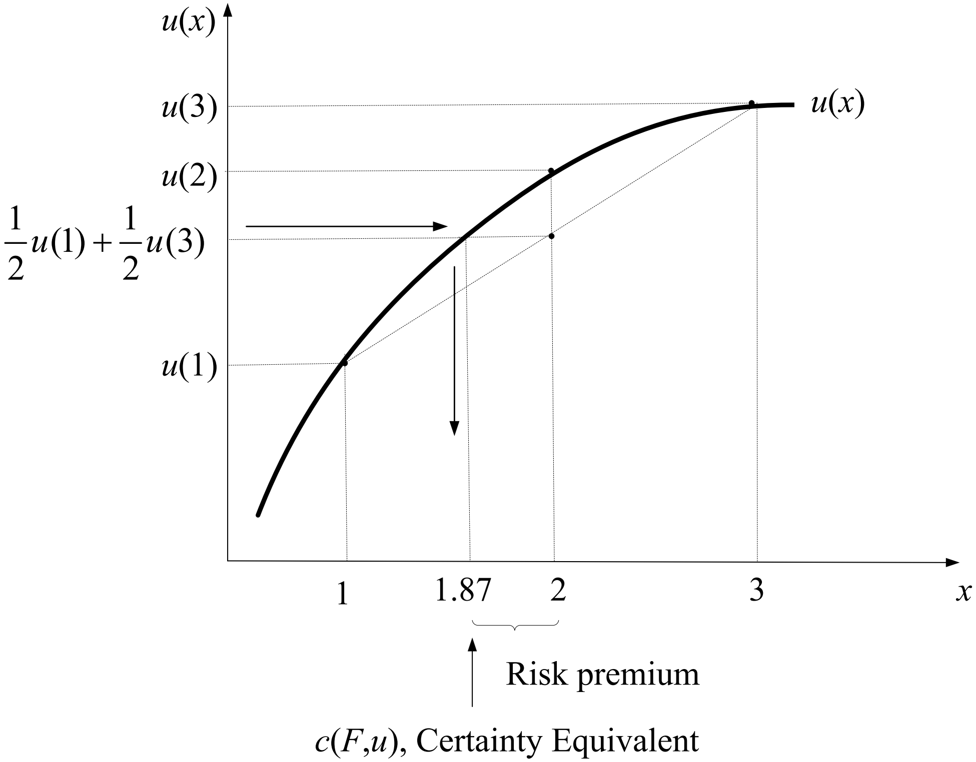
Measuring Risk Preferences

- Certainty equivalent for a risk-averse individual
  - $c(F,u)$ is the amount of money (x) for which utility is equal to the EU of the lottery

    $$u(c(F,u)) = \frac{1}{2}u(1) + \frac{1}{2}u(3)$$

  - ***Risk premium*** (RP): the amount that a risk-averse person would *pay* to avoid taking a risk:

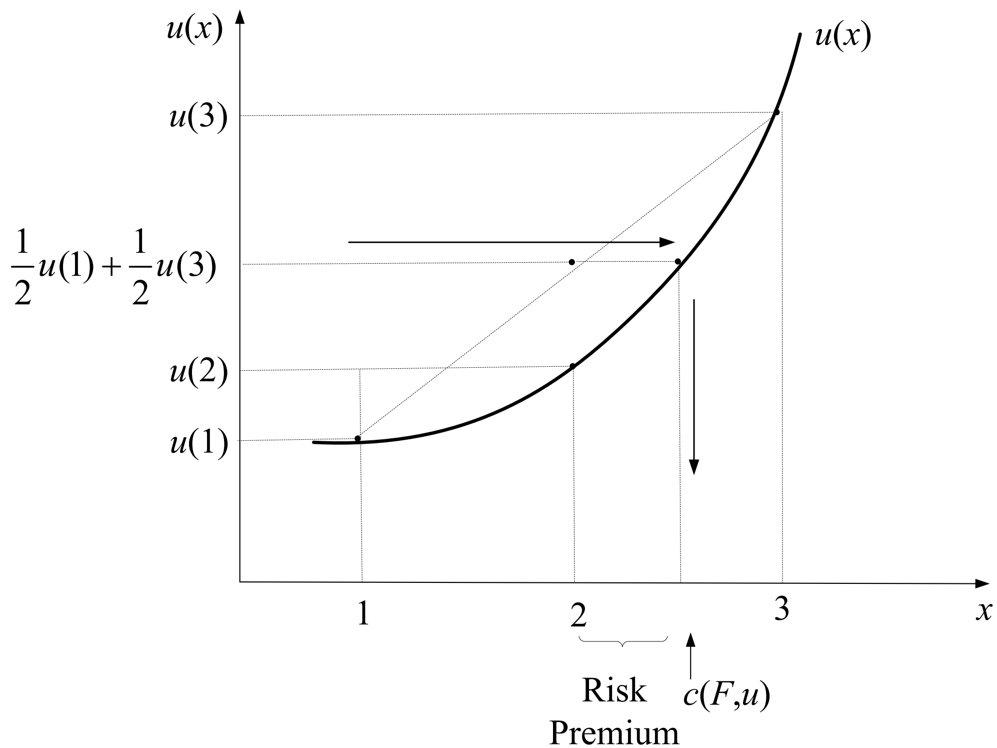
    $$RP = EV - c(F,u) > 0$$



Measuring Risk Preferences

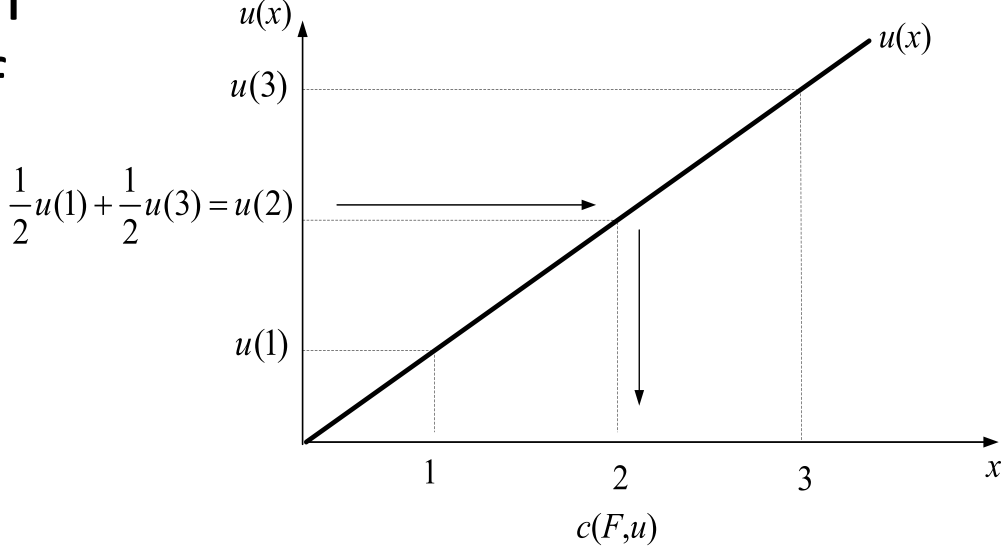
- Certainty equivalent for a risk lover
  - Individual would have to be given an amount of money *above* the expected value of the lottery in order to convince him to "stop playing" the lottery:

$$RP = EV - c(F, u) < 0$$



Measuring Risk Preferences

- Certainty equivalent for a risk neutral individual
  - The certainty equivalent $c(F,u)$ coincides with the expected value of the lottery.
  - Hence,
    $RP = EV - c(F,u) = 0$



Measuring Risk Preferences

- ***Probability premium***, $\pi(x, \varepsilon, u)$:
  - An alternative measure of risk aversion
  - It is the excess in winning probability over fair odds that makes the individual indifferent between the certainty outcome x and a gamble between the two outcomes $x + \varepsilon$ and $x - \varepsilon$:

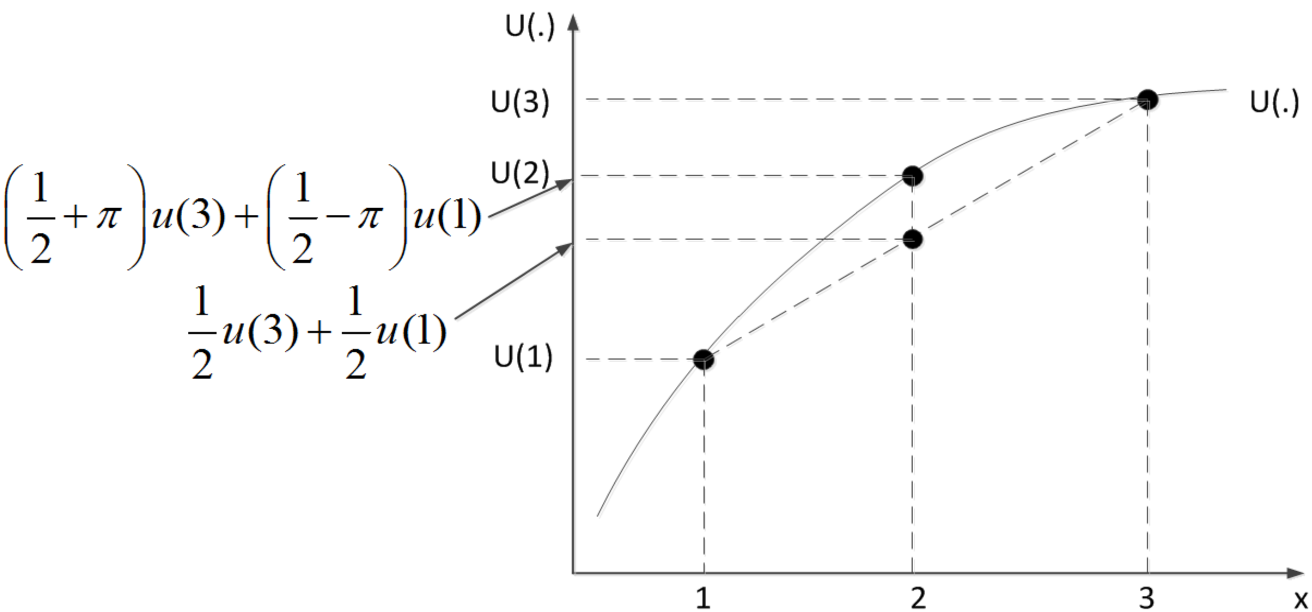
$$u(x) = \left[\frac{1}{2} + \pi(x, \varepsilon, u)\right] u(x + \varepsilon) + \left[\frac{1}{2} - \pi(x, \varepsilon, u)\right] u(x - \varepsilon)$$

  - *Intuition*: Better than fair odds must be given for the individual to accept the risk.

Measuring Risk Preferences

- The "extra probability" π that is needed to make the EU of the lottery coincides with the utility of the expected lottery:

$$u(2) = \left[\frac{1}{2} + \pi\right] u(3) + \left[\frac{1}{2} - \pi\right] u(1)$$



Measuring Risk Preferences

- The following properties are equivalent:
  1) The decision maker is risk averse.
  2) The Bernoulli utility function $u(x)$ is concave, $u''(x) \leq 0$.
  3) The certainty equivalent is lower than the expected value of the lottery, i.e., $c(F,u) \leq \int u(x)dF(x)$.
  4) The risk premium is positive, $RP = EV - c(F,u)$.
  5) The probability premium is positive for all x and ε, i.e., $\pi(x,\varepsilon,u) \geq 0$.

Measuring Risk Preferences

- ***Arrow-Pratt coefficient of absolute risk aversion***:

$$r_A(x) = -\frac{u''(x)}{u'(x)}$$

  – Clearly, the greater the curvature of the utility function, $u''(x)$, the larger the coefficient $r_A(x)$.

  – But, why do not we simply have $r_A(x) = u''(x)$?

    - Because it will not be invariant to positive linear transformations of the utility function, such as $v(x) = \beta u(x)$. That is, $v''(x) = \beta u''(x)$ is affected by the transformation, but the above coefficient of risk aversion is unaffected.

$$r_A(x) = -\frac{\beta u''(x)}{\beta u'(x)} = -\frac{u''(x)}{u'(x)}$$

Measuring Risk Preferences

- ***Example*** (CARA utility function).
  - Take $u(x) = -e^{-ax}$ where $a > 0$. Then
  $$r_A(x) = -\frac{u''(x)}{u'(x)} = -\frac{-a^2e^{-ax}}{ae^{-ax}} = a$$
  which is constant in wealth x.
  - The literature refers to this Bernoulli utility function as the *Constant Absolute Risk Aversion* (CARA).

Measuring Risk Preferences

- If $r_A(x)$ decreases as we increase wealth x, then we say that such Bernoulli utility function satisfies *decreasing absolute risk aversion* (DARA)

$$\frac{\partial r_A(x)}{\partial x} < 0$$

- *Intuition*: wealthier people are willing to bear more risk than poorer people. Note, however, that this is NOT due to different utility functions, but because the same utility function is evaluated at higher/lower wealth levels.
- A sufficient condition for DARA is $u'''(x) > 0$.

Measuring Risk Preferences

- ***Arrow-Pratt coefficient of relative risk aversion***:

$$r_R(x) = -x \cdot \frac{u''(x)}{u'(x)} \quad \text{or} \quad r_R(x) = x \cdot r_A(x)$$

  - $r_R(x)$ does not vary with the wealth level at which it is evaluated.
  - We can show that

$$\frac{\partial r_R(x)}{\partial x} = \underbrace{r_A(x)}_{+} + x \cdot \frac{\partial r_A(x)}{\partial x}$$

  - Therefore,

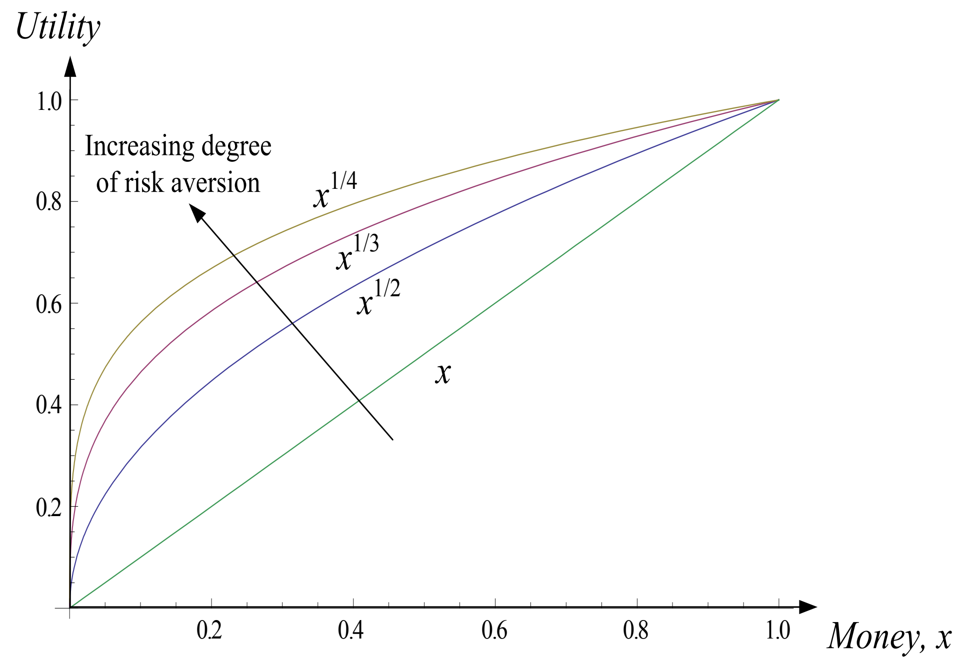
$$\frac{\partial r_R(x)}{\partial x} < 0 \underset{\nLeftarrow}{\Rightarrow} \frac{\partial r_A(x)}{\partial x} < 0$$

Measuring Risk Preferences

- ***Example***:
  – Take $u(x) = x^b$. Then
  $$r_R(x) = -x \cdot \frac{b(b-1)x^{b-2}}{bx^{b-1}} = 1 - b$$
  for all x.
  – The literature refers to this Bernoulli utility function as the *Constant Relative Risk Aversion* (CRRA).

Measuring Risk Preferences

- ***Example*** (continued):
  – Consider a CRRA utility function $u(x) = x^b$ for $b = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.
  – $r_R(x)$ increases, respectively, to $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, making utility function more concave.



Measuring Risk Preferences

- A utility function $u_A(\cdot)$ exhibits more *strong risk aversion* than another utility function $u_B(\cdot)$ if, there is a constant $\lambda > 0$,

$$\frac{u_A''(x_1)}{u_B''(x_1)} \geq \lambda \geq \frac{u_A'(x_2)}{u_B'(x_2)}$$

- In addition, if $x_1 = x_2$, the above condition can be re-written as

$$\frac{u_A''(x_1)}{u_A'(x_1)} \geq \frac{u_B''(x_1)}{u_B'(x_1)}$$

- Then, $u_A(\cdot)$ also exhibits more risk aversion than $u_B(\cdot)$.

Measuring Risk Preferences

- For two utility functions u_1 and u_2, where u_2 is a concave transformation of u_1, the following properties are equivalent:
  1) There exists an increasing concave function $\varphi(\cdot)$ such that $u_2(x) = \varphi(u_1(x))$ for any x. That is, $u_2(\cdot)$ is more concave than $u_1(\cdot)$.
  2) $r_A(x, u_2) \geq r_A(x, u_1)$ for any x.
  3) $c(F, u_2) \leq c(F, u_1)$ for any lottery $F(\cdot)$.
  4) $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε.

Measuring Risk Preferences

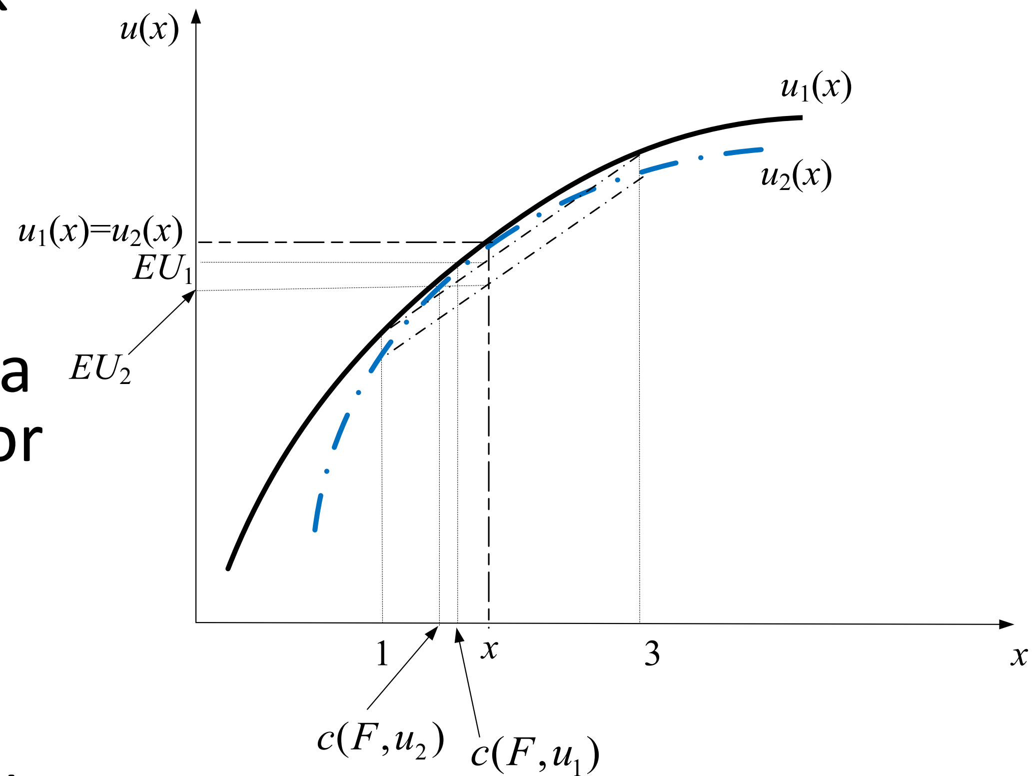
5) Whenever $u_2(\cdot)$ finds a lottery $F(\cdot)$ at least as good as a riskless outcome $\bar{x}$, then $u_1(\cdot)$ also finds such a lottery $F(\cdot)$ at least as good as $\bar{x}$. That is

$$EU_2 = \int u_2(x)dF(x) \geq u_2(\bar{x}) \Longrightarrow$$

$$EU_1 = \int u_1(x)dF(x) \geq u_1(\bar{x})$$

Measuring Risk Preferences

- Different degrees of risk aversion
- $u_1(\cdot)$ and $u_2(\cdot)$ are evaluated at the same wealth level x.
- The same lottery yields a larger expected utility for the individual with *less risk averse* preferences, $EU_1 > EU_2$.
- $c(F, u_2) < c(F, u_1)$, reflecting that individual 2 is more risk averse.



Prospect Theory and Reference-Dependent Utility

Prospect Theory

- ***Prospect theory***: a decision maker's total value from a list of possible outcomes $x = (x_1, x_2, \dots, x_n)$ with associated probabilities $p = (p_1, p_2, \dots, p_n)$ is

$$v(x,p) = \sum_{i=1}^{n} w(p_i) \cdot v(x_i)$$

where
  - $w(p_i)$ is a "probability weighting function"
  - $v(x_i)$ is the "value function" the individual obtains from outcome x_i

Prospect Theory

- **Three main differences relative to standard expected utility theory**:
- First, $w(p_i) \neq p_i$:
  - if $w(p_i) > p_i$, individuals *overestimate* the likelihood of outcome x_i
  - if $w(p_i) < p_i$, individuals *underestimate* the likelihood of outcome x_i
  - if $w(p_i) = p_i$, the model coincides with standard expected utility theory.

Prospect Theory

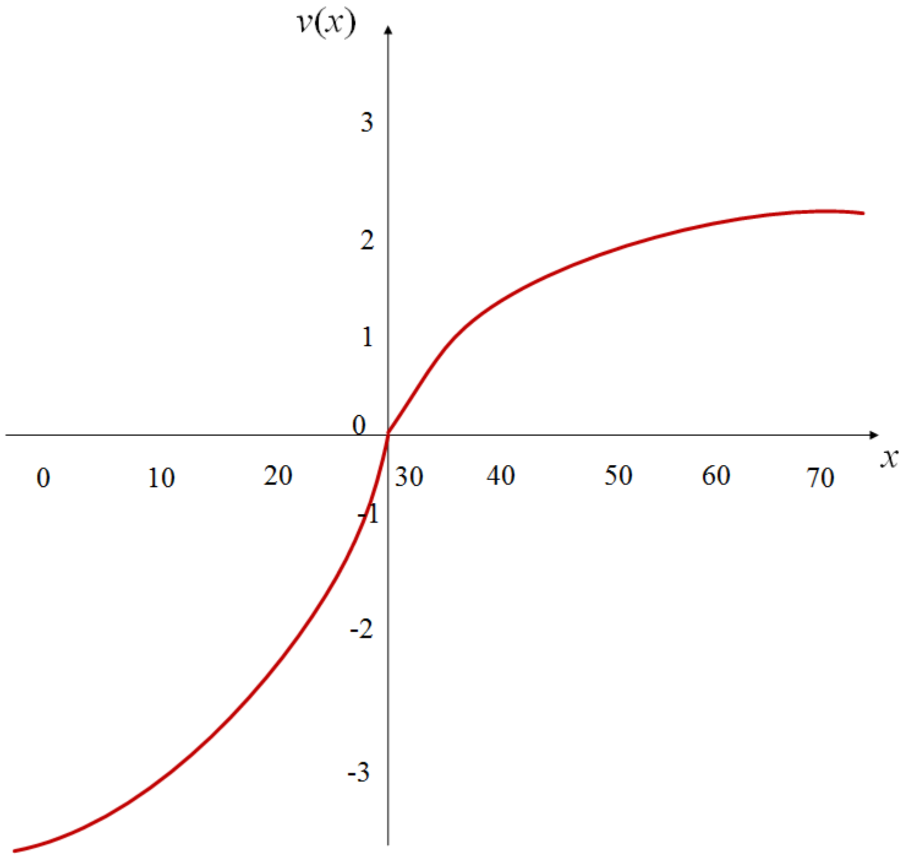
- Second, every payoff x_i is evaluated relative to a "reference point" x_0, with the value function $v(x_i)$, which is
  - Increasing and concave, $v''(x_i) < 0$, for all $x_i > x_0$,
    - That is, the individual is risk averse for gains.
  - Decreasing and convex, $v''(x_i) > 0$, for all $x_i < x_0$
    - That is, the individual is risk lover for losses.
  - *Extremes*:
    - if $x_0 = 0$, the individual is risk averse for all payoffs;
    - if $x_0 = +\infty$, he is risk lover for all payoffs.

Prospect Theory

- Third, value function $v(x_i)$ has a kink at the reference point x_0.
  - The curve becomes steeper for losses (to the left of x_0) than for gains (to the right of x_0).
    - Loss aversion:
    - A given loss of $\$a$ produces a larger disutility than a gain of the same amount.

Prospect Theory

- Value function in prospect theory



Prospect Theory

- ***Example***:
  - Consider as in Tversky and Kahneman (1992)

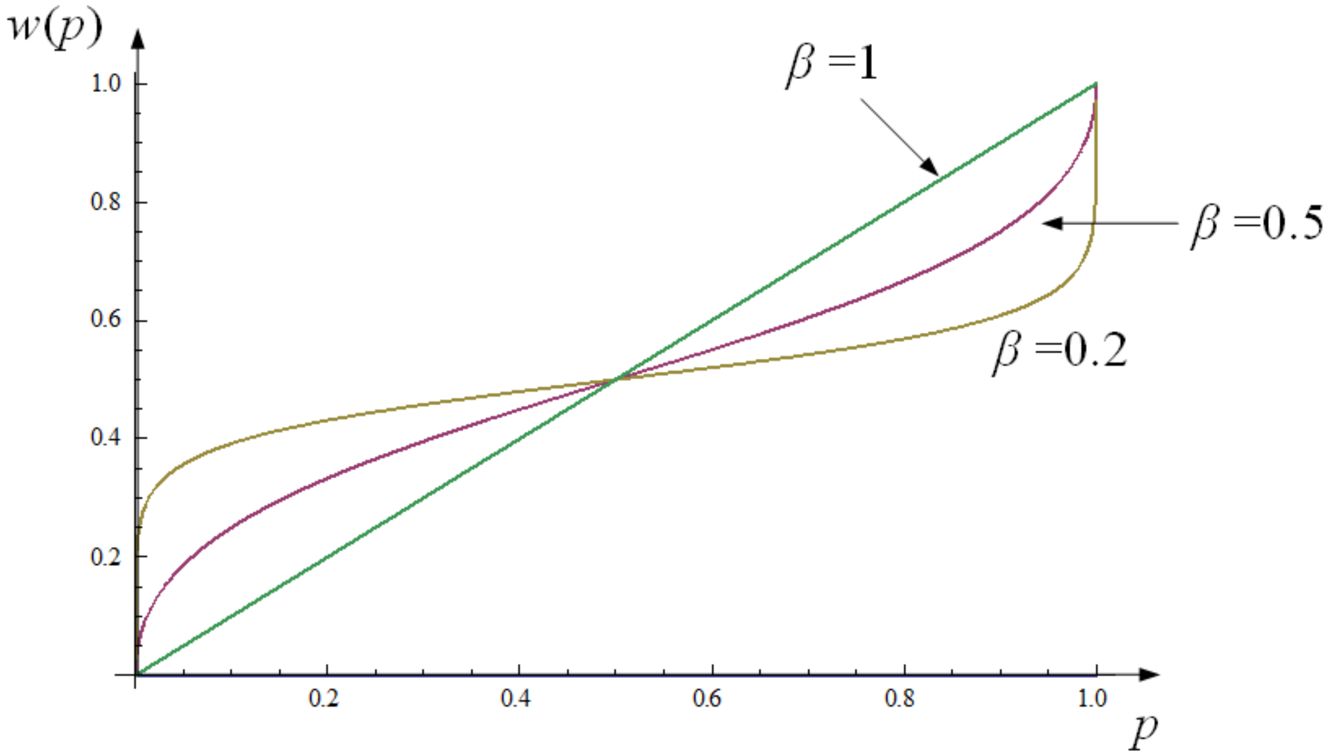
$$w(p) = \frac{p^{\beta}}{\left[p^{\beta}+(1-p)^{\beta}\right]^{\frac{1}{\beta}}} \quad \text{and} \quad v(x) = x^{\alpha}$$

where $0 < \beta < 1$, and $0 < \alpha < 1$.

- Note that this implies probability weighting, but does not consider a value function with loss aversion relative to a reference point.

Prospect Theory

- ***Example*** (continued):
  - Depicting the probability weighting function



Prospect Theory

- ***Example***:
  - A common value function is

$$v(x_i) = {x_i}^{\alpha} \quad \text{if } x_i \geq x_0\text{, and}$$
$$= -\lambda(-x_i)^{\alpha} \quad \text{if } x_i < x_0$$

  where $0 < \alpha \leq 1$, and $\lambda \geq 1$ represents loss aversion.

    - If $\lambda = 1$ the individual does not exhibit loss aversion.

Prospect Theory

- ***Example***:
  - Common simplifications, assume $\alpha = \beta = 1$ (which implies no probability weighting, and linear value functions), to estimate λ.
  - Average estimates $\lambda = 2.25$ and $\beta = 0.88$

Prospect Theory

- Further reading:
  - Nicholas Barberis (2013) "Thirty Years of Prospect Theory in Economics: A Review and Assessment," *Journal of Economic Perspectives*, 27(1), pp. 173-96.
  - R. Duncan Luce and Peter C. Fishburn (1991) "Rank and sign-dependent linear utility models for binary gambles." *Journal of Economic Theory*, 53, pp. 75–100.
  - Daniel Kahneman and Amos Tversky (1992) "Advances in prospect theory: Cumulative representation of uncertainty" *Journal of Risk and Uncertainty*, 5(4), pp. 297–323.
  - Peter Wakker and Amos Tversky (1993) "An axiomatization of cumulative prospect theory." *Journal of Risk and Uncertainty*, 7, pp. 147–176.

Reference-Dependent Utility

- Individual preferences are affected by *reference points*. Thus, gains and loses can be evaluated differently.
- Consider a consumption vector $x \in \mathbb{R}^n$ which is evaluated against a n-dimensional reference vector $r \in \mathbb{R}^n$. Utility function is

$$u(x|r) = m(x) + n(x|r)$$

where $n(x_k|r_k) = \mu\big(m_k(x_k)\big) - m_k(r_k)$ measures the gain/loss of consuming x_k units of good k relative to its reference amount r_k.

Reference-Dependent Utility

- For lotteries with cumulative distribution function $F(x)$,

$$U(F|r) = \int u(x|r)dF(x)$$

- For lotteries over the set of reference points

$$u(F|G) = \int \int u(x|r)dG(r)dF(x)$$

Reference-Dependent Utility

- Further reading:
  - "Reference-Dependent Consumption Plans" (2009) by Koszegi and Rabin, *American Economic Review*, vol. 99(3).
  - "Rational Choice with Status Quo Bias" (2005) by Masatlioglu and Ok, *Journal of Economic Theory,* vol. 121(1).
  - "On the complexity of rationalizing behavior" (2007) Apesteguia and Ballester, *Economics Working Papers* 1048.

Comparison of Payoff Distributions

Comparison of Payoff Distributions

- So far we compared utility functions, but not the distribution of payoffs.
- Two main ideas:
  1) $F(\cdot)$ yields unambiguously *higher returns* than $G(\cdot)$. We will explore this idea in the definition of first order stochastic dominance (FOSD);
  2) $F(\cdot)$ is unambiguously *less risky* than $G(\cdot)$. We will explore this idea in the definition of second order stochastic dominance (SOSD).

Comparison of Payoff Distributions

- ***FOSD***: $F(\cdot)$ FOSD $G(\cdot)$ if, for every non-decreasing function $u: \mathbb{R} \to \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

- The distribution of monetary payoffs $F(\cdot)$ FOSD the distribution of monetary payoffs $G(\cdot)$ if and only if

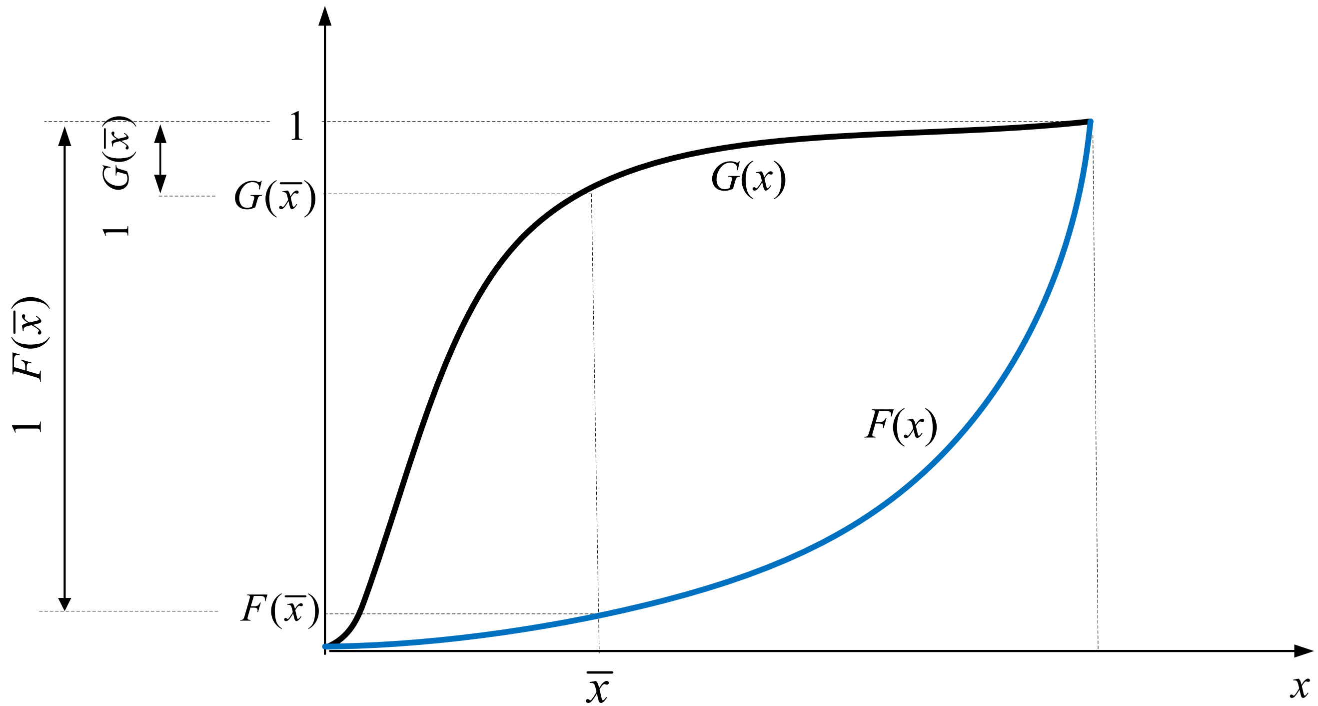
$$F(x) \leq G(x) \text{ or } 1 - F(x) \geq 1 - G(x)$$

for every x.

- *Intuition*: For every amount of money x, the probability of getting at least x is higher under $F(\cdot)$ than under $G(\cdot)$.

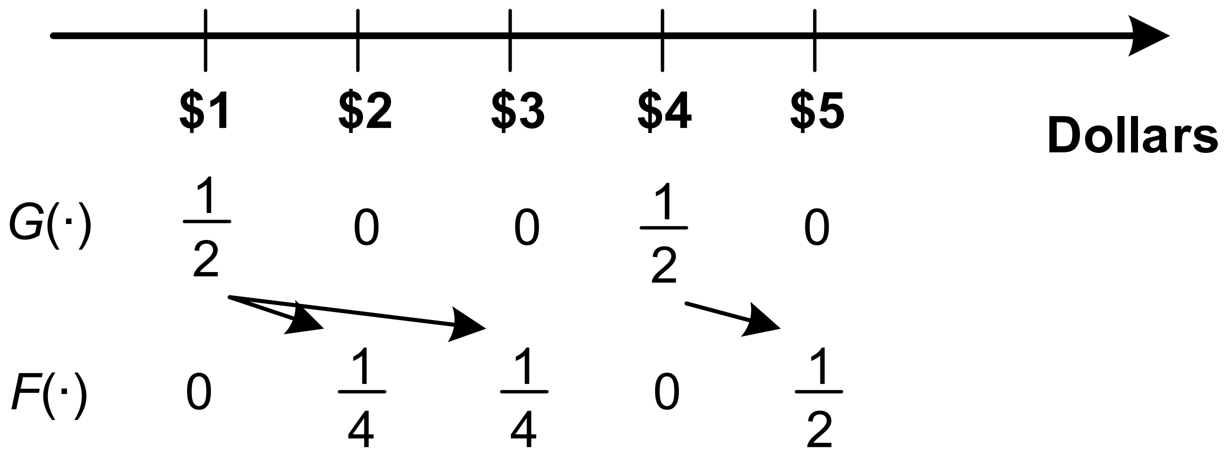
Comparison of Payoff Distributions

- At any given outcome x, the probability of obtaining prizes above x is higher with lottery $F(\cdot)$ than with lottery $G(\cdot)$, i.e., $1 - F(x) \geq 1 - G(x)$.



Comparison of Payoff Distributions

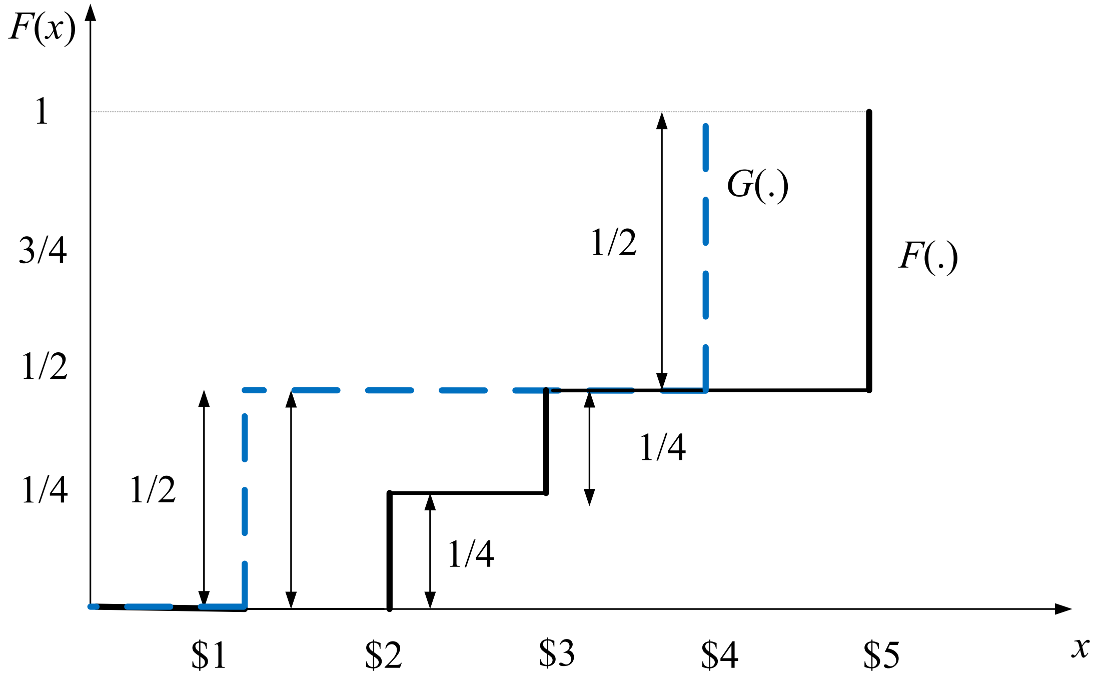
- ***Example***:
  - Let us take lotteries $F(\cdot)$ and $G(\cdot)$ over discrete outcomes.



How can we know if $F(\cdot)$ FOSD $G(\cdot)$?

Comparison of Payoff Distributions

- ***Example*** (continued):
  - $F(\cdot)$ lies below lottery $G(\cdot)$. Hence, $F(\cdot)$ concentrates more probability weight on higher monetary outcomes.
  - Thus, $F(\cdot)$ FOSD $G(\cdot)$.

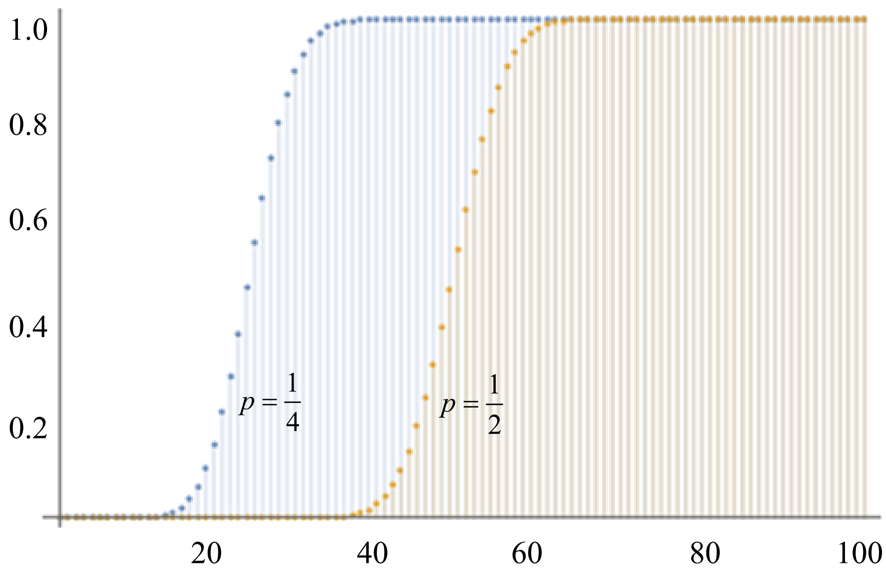


Comparison of Payoff Distributions

- ***Example*** (Binomial distribution):
  - Consider the binomial distribution

$$F(x; N, p) = \binom{N}{p} p^x (1-p)^{N-x}$$

  - where $x \in [0, N]$. Assuming $N = 100$ and parameter p increasing from $p = \frac{1}{4}$ to $p = \frac{1}{2}$. Then, $F(x; 100,1/2)$ FOSD $F(x; 100,1/4)$.



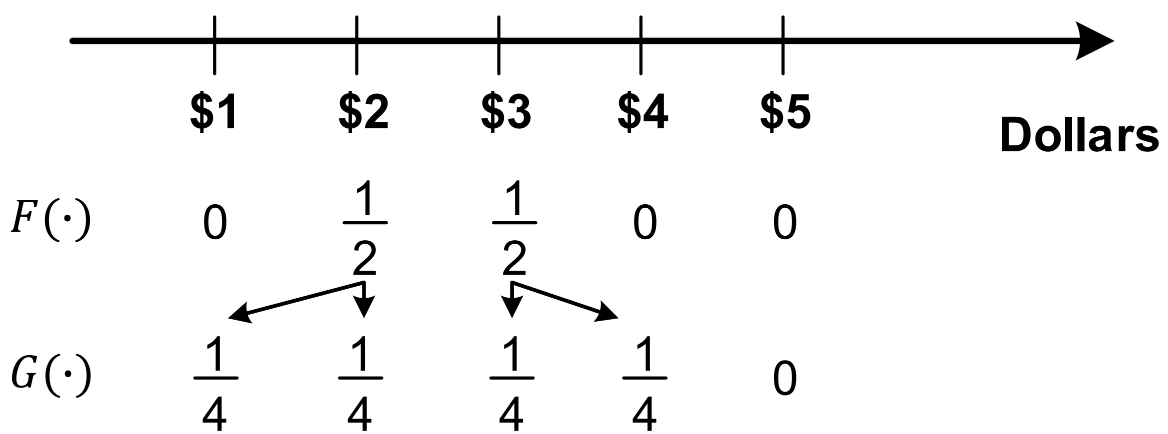
Comparison of Payoff Distributions

- We now focus on the *riskiness* or *dispersion* of a lottery, as opposed to higher/lower returns of lottery (FOSD).
- To focus on riskiness, we assume that the CDFs we compare have the *same mean* (i.e., same expected return).
- ***SOSD***: $F(\cdot)$ SOSD $G(\cdot)$ if, for every non-decreasing function $u: \mathbb{R} \to \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

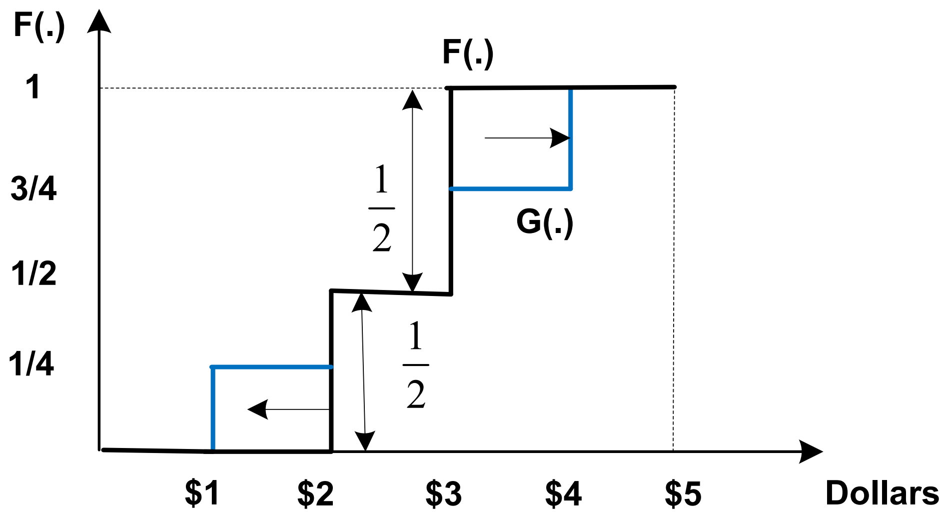
Comparison of Payoff Distributions

- ***Example*** (Mean-Preserving Spread):
  - Let us take lotteries $F(\cdot)$ and $G(\cdot)$ over discrete outcomes.
  - Lottery $G(\cdot)$ spreads the probability weight of lottery $F(\cdot)$ over a larger set of monetary outcomes.
  - The mean is nonetheless unaltered (2.5).
  - For these two reasons, we say that a CDF is a mean preserving spread of the other.



Comparison of Payoff Distributions

- $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$, but it is riskier than $F(\cdot)$ in the SOSD sense.
- Note that neither FOSD the other
  - $F(\cdot)$ is not above/below $G(\cdot)$ for all x



Comparison of Payoff Distributions

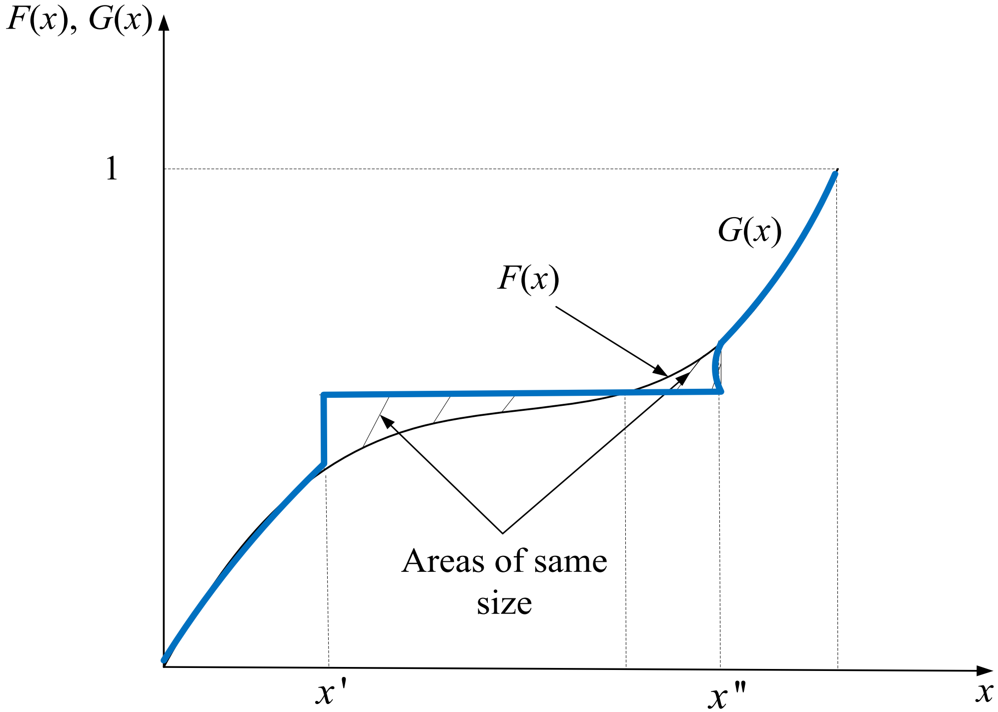
- ***Example*** (Elementary increase in risk):
  - $G(\cdot)$ is an *Elementary Increase in Risk* (EIR) of another CDF $F(\cdot)$ if $G(\cdot)$ takes all the probability weight of an interval $[x', x'']$ and transfers it to the *end points* of this interval, x' and x'', such that the mean of the original lottery is preserved.
  - EIR is a mean-preserving spread (MPS), but the converse is not necessarily true:

$$EIR \underset{\not\Leftarrow}{\overset{\Rightarrow}{}} MPS$$

  - Hence, if $G(\cdot)$ is an EIR of $F(\cdot)$, then $F(\cdot)$ SOSD $G(\cdot)$.

Comparison of Payoff Distributions

- ***Example*** (continued):
  - both CDFs $F(\cdot)$ and $G(\cdot)$ maintain the same mean.
  - $G(\cdot)$ concentrates more probability at the end points of the interval $[x', x'']$ than $F(\cdot)$.



Comparison of Payoff Distributions

- ***Hazard rate dominance***: The hazard rate of lottery $F(x)$ is

$$HR_F(x) = \frac{f(x)}{1 - F(x)}$$

  - *Intuition*: It measures the instantaneous probability of an event happening at time x given that it did not happen before x.
  - *Example*: a computer stops working at exactly x
  - If $HR_F(x) \leq HR_G(x)$, lottery $F(x)$ dominates $G(x)$ in terms of the hazard rate.

Comparison of Payoff Distributions

– Since $-HR_F(x)$ can be expressed as

$$-HR_F(x) = \frac{d}{dx}\ln(1 - F(x))$$

– Solving for $F(x)$,

$$F(x) = 1 - \exp\left(-\int_0^x HR_F(t)dt\right)$$

– Then,

$$F(x) = 1 - \exp\left(-\int_0^x HR_F(t)dt\right)$$

$$\leq 1 - \exp\left(-\int_0^x HR_F(t)dt\right) = G(x)$$

– Thus, $HR_F(x) \leq HR_G(x)$ implies that $F(x)$ FOSD $G(x)$.

Comparison of Payoff Distributions

- ***Reverse hazard rate***: The reverse hazard rate of lottery $F(x)$ is

$$RHR_F(x) = \frac{f(x)}{F(x)}$$

  - *Intuition*: It measures the probability that, conditional on the realized payoff in the lottery being equal or lower than x, the payoff you receive is exactly x.
  - If $RHR_F(x) \geq RHR_G(x)$, lottery $F(x)$ dominates $G(x)$ in terms of the reverse hazard sense.

Comparison of Payoff Distributions

– Let us express $RHR_F(x)$ as

$$RHR_F(x) = \frac{d}{dx}\ln(F(x))$$

– Solving for $F(x)$,

$$F(x) = \exp\left(-\int_0^x RHR_F(t)dt\right)$$

– Then,

$$F(x) = \exp\left(-\int_0^x RHR_F(t)dt\right) \leq \exp\left(-\int_0^x RHR_F(t)dt\right) = G(x)$$

– Thus, $RHR_F(x) \geq RHR_G(x)$ implies that $F(x)$ FOSD $G(x)$.

Comparison of Payoff Distributions

- ***Likelihood ratio***: The likelihood ratio of a lottery $F(x)$ is

$$LR_F = \frac{f(y)}{f(x)}$$

for any two payoffs x and y, where $y > x$.

  – $F(x)$ dominates $G(x)$ in terms of likelihood ratio if

$$\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)}$$

Comparison of Payoff Distributions

- LR dominance implies HR dominance:
  - Let us rewrite LR dominance as
$$\frac{g(y)}{g(x)} \leq \frac{f(y)}{f(x)}$$
  - Then, for all x,
$$\int_0^\infty \frac{g(y)}{g(x)} dy \leq \int_0^\infty \frac{f(y)}{f(x)} dy$$
  - Simplifying
$$\frac{1-G(x)}{g(x)} \leq \frac{1-F(x)}{f(x)} \text{ or } \frac{f(x)}{1-F(x)} \leq \frac{g(x)}{1-G(x)}$$
which implies $HR_F(x) \leq HR_G(x)$.

Comparison of Payoff Distributions

- Summary:
  - LR dominance implies HR dominance
  - HR and RHR dominance imply FOSD.

Appendix 5.1:
State-Dependent Utility

State-Dependent Utility

- So far the decision maker only cared about the payoff arising from every outcome of the lottery.
- Now we assume that the decision maker cares not only about his monetary outcomes, but also about the ***state of nature*** that causes every outcome.
  - That is, $u_{\text{state 1}}(x) \neq u_{\text{state 2}}(x)$ for given x.

State-Dependent Utility

- Let us assume that each of the possible monetary payoffs in a lottery is generated by an underlying cause (i.e., an underlying state of nature).
- ***Examples***:
  - The monetary payoff of an insurance policy is generated by a car accident
    - State of nature = {car accident, no car accident}
  - The monetary payoff of a corporate stock is generated by the state of the economy
    - State of nature = {economic growth, economic depression}

State-Dependent Utility

- Generally, let $s \in S$ denote a state of nature, where S is a finite set.
- Every state s has a well-defined, objective probability $\pi_s \geq 0$.
- A random variable is function $g: S \rightarrow \mathbb{R}$, that maps states into monetary payoffs.

State-Dependent Utility

- ***Examples*** (revisited):
  - *Car accident*: the random variable assigns a monetary value to the state of nature care accident, and to the state of nature no accident.

State of nature	Probability	Monetary payoff
Car accident	π_{accident}	Damage + Deductible – Premium = $1,000
No car accident	$\pi_{\text{no accident}}$	Premium = -$50

State-Dependent Utility

- ***Examples*** (revisited):
  - *Corporate stock*: the random variable assigns a monetary value to the state of nature econ. growth, and to the state of nature eco. depression.

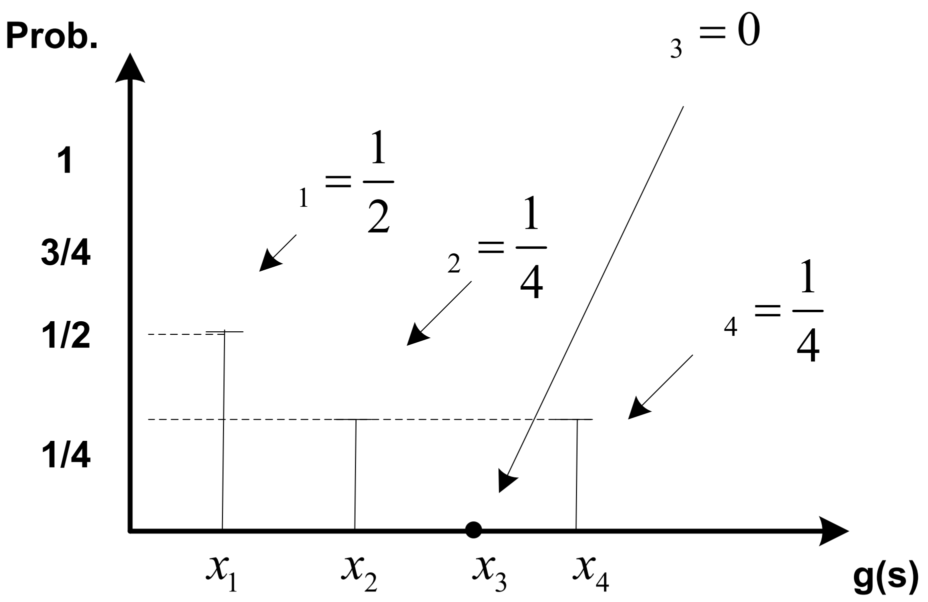
State of nature	**Probability**	**Monetary payoff**
Economic growth	π_{growth}	Dividends, higher price of shares = \$250
Economic depression	$\pi_{\text{depression}}$	No dividends, loss if we sell shares = -\$125

State-Dependent Utility

- Every random variable $g(\cdot)$ can be used to represent lottery $F(\cdot)$ over monetary payoffs as
$$F(x) = \sum_{\{s:\, g(s) \leq x\}} \pi_s$$
where $\{s:\ g(s) \leq x\}$ represents all those states of nature s that generate a monetary payoff $g(s) \in \mathbb{R}$ below a cutoff payoff x.
- The random variable $g(\cdot)$ generates a monetary payoff for every state of nature $s \in S$, and since S is finite, we can represent this list of monetary payoffs as
$$(x_1, x_2, \dots, x_S) \in \mathbb{R}_+^S$$
where x_s is the monetary payoff corresponding to state of nature s.

State-Dependent Utility

- ***Example***:
  - A random variable $g(\cdot)$ describes the monetary outcome associated to the four states of nature $S = \{1,2,3,4\}$.
  - Outcomes are ordered from lower to higher $x_1 \le x_2 \le x_3 \le x_4$.



State-Dependent Utility

- ***Example*** (continued):
  - Hence,

$$F(x_1) = \pi_1 = \frac{1}{2}$$
$$F(x_2) = \pi_1 + \pi_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$F(x_3) = \pi_1 + \pi_2 + \pi_3 = \frac{1}{2} + \frac{1}{4} + 0 = \frac{3}{4}$$
$$F(x_4) = \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

- Disadvantage of $F(x)$:
  - For a given x, we cannot keep track of which state(s) of nature generated x.

State-Dependent Utility: Extended EU representation

- We now have a preference relation $\succsim$ ranks lists of monetary payoffs $(x_1, x_2, \ldots, x_S) \in \mathbb{R}_+^S$.
- Note the similarity of this setting with that in consumer theory:
  - Preferences over bundles then, preferences over lists of monetary payoffs here.
  - Since $(x_1, x_2, \ldots, x_S) \in \mathbb{R}_+^S$ specifies one payoff for each state of nature, this list is referred to as *contingent commodities*.

State-Dependent Utility: Extended EU representation

- Preference relation $\succsim$ has an ***Extended EU representation*** if for every $s \in S$, there is a function $u_S: \mathbb{R}_+ \to \mathbb{R}_+^S$ (mapping the monetary outcome of state s, x_s, into a utility value in $\mathbb{R}$), such that for any two lists of monetary outcomes $(x_1, x_2, \ldots, x_S) \in \mathbb{R}_+^S$ and $(x_1', x_2', \ldots, x_S') \in \mathbb{R}_+^S$,

$$(x_1, x_2, \ldots, x_S) \succsim (x_1', x_2', \ldots, x_S') \text{ iff}$$

$$\sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x_s')$$

- The main difference with the previous sections is that now the Bernoulli utility function is *state-dependent*, $u_s(\cdot)$, whereas in the previous sections it was *state-independent*, $u(\cdot)$.

State-Dependent Utility: Extended EU representation

- Graphical representation:
  - First, at the "certainty line" the decision maker receives the same monetary amount, regardless the state of nature, $x_1 = x_2$.
  - Second, all the (x_1, x_2) pairs on a given ind. curve satisfy $\pi_1 \cdot u_1(x_1) + \pi_2 \cdot u_2(x_2) = \bar{U}$
  - Third, the upper contour set of an ind. curve that passes through point $(\bar{x}_1, \bar{x}_2)$ satisfy

$$\pi_1 \cdot u_1(x_1) + \pi_2 \cdot u_2(x_2)$$
$$\geq \pi_1 \cdot u_1(\bar{x}_1) + \pi_2 \cdot u_2(\bar{x}_2)$$

or, more generally, $\sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(\bar{x}_s)$.

State-Dependent Utility: Extended EU representation

- Graphical representation:
  - Fourth, movement along a given ind. curve does not change the decision maker's utility level. Hence, totally differentiating

$$\pi_1 \cdot \frac{\partial u_1(\bar{x}_1)}{\partial x_1} dx_1 + \pi_2 \cdot \frac{\partial u_2(\bar{x}_2)}{\partial x_2} dx_2 = 0$$

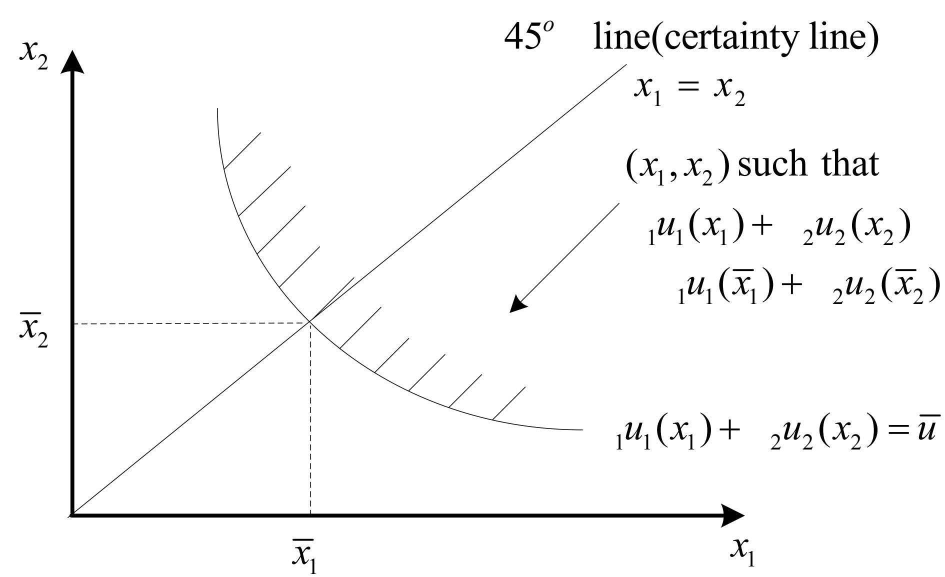
    and re-arranging,

$$\frac{dx_2}{dx_1} = -\frac{\pi_1 \cdot \frac{\partial u_1(\bar{x}_1)}{\partial x_1}}{\pi_2 \cdot \frac{\partial u_2(\bar{x}_2)}{\partial x_2}} = -\frac{\pi_1 \cdot u_1'(\bar{x}_1)}{\pi_2 \cdot u_2'(\bar{x}_2)}$$

    which represents the slope of the ind. curve, evaluated at point $(\bar{x}_1, \bar{x}_2)$. This is really similar to MRS.

State-Dependent Utility: Extended EU representation

- Graphical representation:
  - The slope of the ind. curve at $(\bar{x}_1, \bar{x}_2)$ is
    $$\frac{dx_2}{dx_1} = -\frac{\pi_1 \cdot u_1'(\bar{x}_1)}{\pi_2 \cdot u_2'(\bar{x}_2)}$$
  - If the Bernoulli utility is state-independent, i.e., $u_1(\cdot) = u_2(\cdot) = \cdots = u_S(\cdot)$, then the slope is
    $$\frac{dx_2}{dx_1} = -\frac{\pi_1}{\pi_2}$$

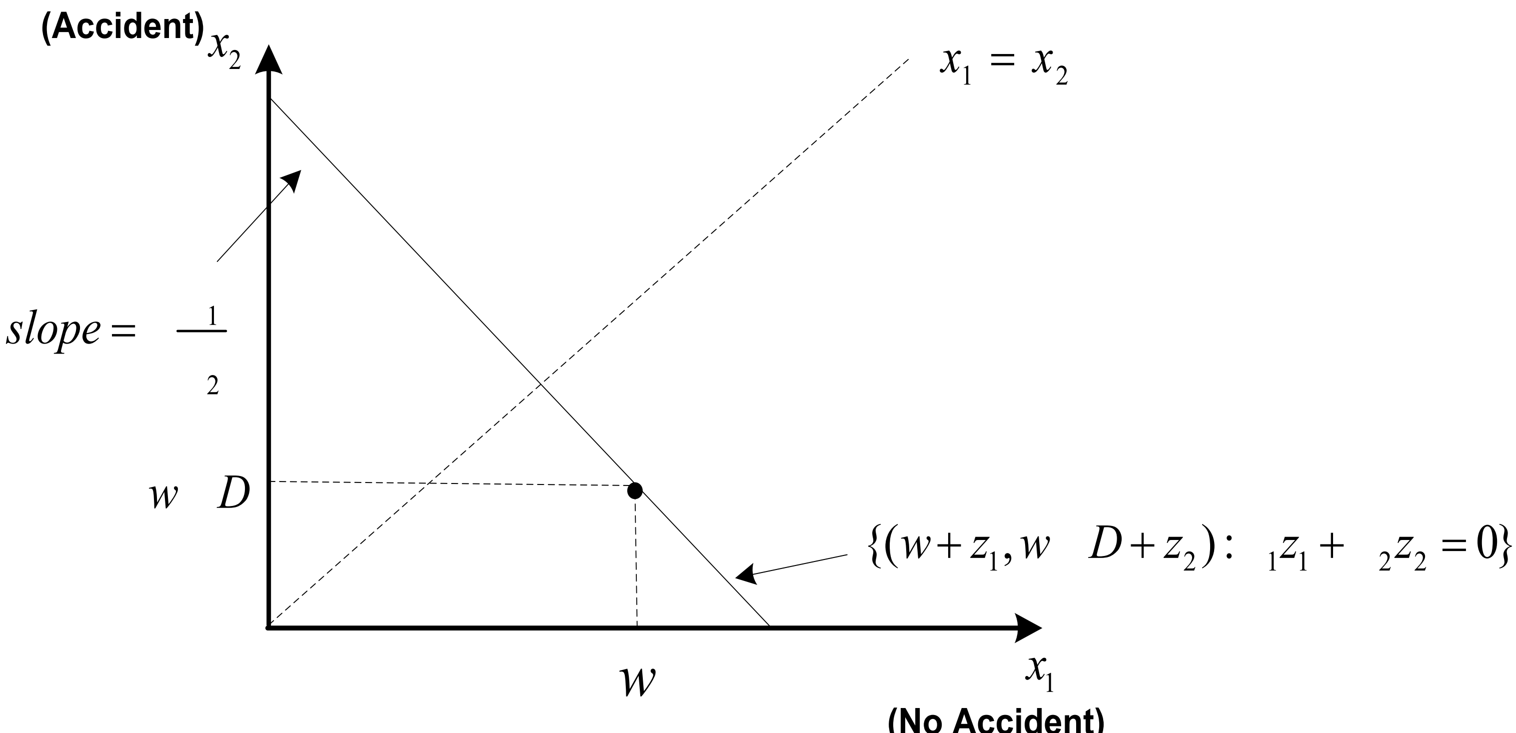


State-Dependent Utility: Extended EU representation

- ***Example*** (Insurance with state-dependent utility):
  - Start from an initial situation of $(w, w - D)$ without insurance, where D is loss from accident.
  - After insurance is purchased, the decision maker gets a payment of z_1 in state 1, and z_2 in state 2, where $z_1 \lesseqgtr 0$ and $z_2 \lesseqgtr 0$,
$$(w + z_1, w - D + z_2)$$
  - Moreover, if the policy is actuarially fair, then its expected payoff is zero,
$$\pi_1 z_1 + \pi_2 z_2 = 0$$

State-Dependent Utility: Extended EU representation

- ***Example*** (continued):
  - The budget line is $z_2 = -\frac{\pi_1}{\pi_2} z_1$



State-Dependent Utility: Extended EU representation

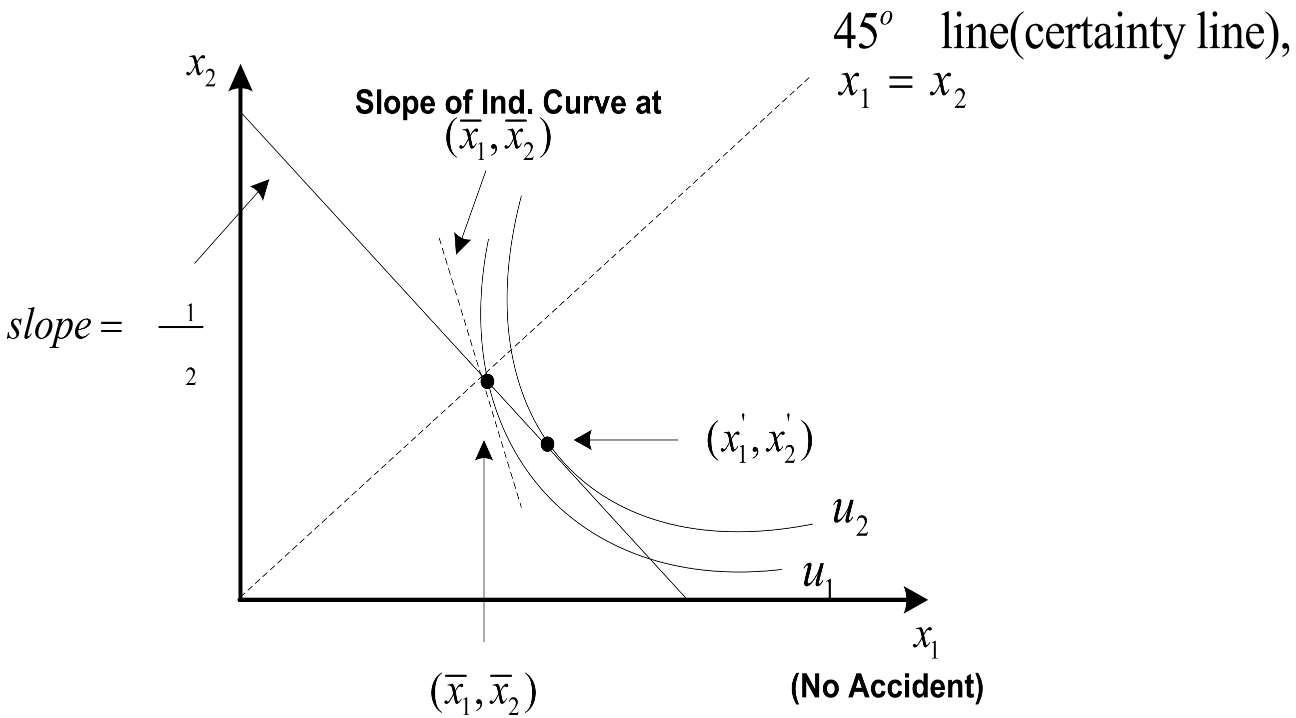
- Without state dependency:
  - Indifference curves are tangent to the budget line at the certainty line, since the slope of the indifference curve is $-\frac{\pi_1}{\pi_2}$.
  - Hence, the decision maker would insure completely since his consumption level is unaffected by the possibility of suffering an accident.

State-Dependent Utility: Extended EU representation

- With state dependency:
  - Indifference curves are NOT tangent to the budget line at the certainty line.
- ***Example*** (continued):
  - The decision-maker prefers a point such as (x_1', x_2') to the certain outcome $(\bar{x}, \bar{x})$.
  - That is, at $(\bar{x}, \bar{x})$ he prefers higher payoffs in state 1 than in state 2 if $u_1'(\bar{x}) > u_2'(\bar{x})$. Otherwise, he would prefer higher payoffs in state 2 than in state 1.

State-Dependent Utility: Extended EU representation

– Note that $u_1'(\bar{x}) > u_2'(\bar{x})$ implies that $\frac{u_1'(\bar{x})}{u_2'(\bar{x})} > 1$ and $-\frac{\pi_1 \cdot u_1'(\bar{x})}{\pi_2 \cdot u_2'(\bar{x})} < -\frac{\pi_1}{\pi_2}$.



State-Dependent Utility: Extended EU representation

- Let us now allow for the possibility that the monetary payoff under state s, x_s, is not a certain amount of money, but a random amount with distribution function $F_s(\cdot)$.
- Hence, all monetary outcomes arising from the S states of world can be described as a lottery $L = (F_1, F_2, \dots, F_S)$.
- Given this "extended" definition of lotteries, we can then re-write the IA, as the "extended" IA.

State-Dependent Utility: Extended EU representation

- ***Extended IA***: The preference relation satisfies the extended IA if, for any three lotteries L, L', and L'' and $\alpha \in (0,1)$, we have that

$$L \succsim L' \text{ iff}$$
$$\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

- Hence, "extended" IA is a mere extension of the standard IA to the case of "extended" lotteries $L = (F_1, F_2, \dots, F_S)$.

State-Dependent Utility: Extended EU representation

- ***Extended EU theorem***: Suppose preferences relation satisfies continuity and the extended IA. Then we can assign a utility function $u_s(\cdot)$ for money in every state s such that for any two lotteries $L = (F_1, F_2, \ldots, F_S)$ and $L' = (F'_1, F'_2, \ldots, F'_S)$ we have

$$L \succsim L' \text{ iff}$$

$$\sum_s \left(\int u_s(x_s) dF_s(x_s) \right) \geq \sum_s \left(\int u_s(x_s) dF'_s(x_s) \right)$$

Appendix 5.2: Subjective Probability Theory

Subjective Probability Theory

- So far we were assuming that probabilities were objective and observable.
- This is not the case in certain cases. Instead people might hold probabilistic *beliefs* about the likelihood of a certain event: ***subjective probability***.

Subjective Probability Theory

- Can we deduce subjective probability from actual behavior? Yes!
- Imagine a decision maker who prefers a gamble

$$(\$1 \text{ in state } 1, \$0 \text{ in state } 2) \succsim (\$0 \text{ in state } 1, \$1 \text{ in state } 2)$$

- If the value of money is the same across states, then he must be assigning a higher subjective probability to state 1 than to state 2.

Subjective Probability Theory

- Let us start with some definitions.
- First, we define state s preferences, $\succsim_s$, on state s lotteries $F_s(\cdot)$ by $F_s(\cdot) \succsim F_s'(\cdot)$ if

$$\int u_s(x_s)dF_s(x_s) \geq \int u_s(x_s)dF_s'(x_s)$$

- Hence, the state preferences $(\succsim_1, \succsim_2, \dots, \succsim_S)$ on state lotteries $(F_1, F_2, \dots, F_S)$ are ***state uniform*** if

$$\succsim_s = \succsim_{s'} \text{ for any two states } s \text{ and } s'$$

Subjective Probability Theory

- That is, preferences over lotteries are state uniform if for any two states s and s', the ranking of any two lotteries $F_s(\cdot)$ and $F_s'(\cdot)$ coincides in both states, i.e.,

$$F_s(\cdot) \succsim F_s'(\cdot) \text{ or}$$

$$F_s'(\cdot) \succsim F_s(\cdot) \text{ or}$$

$$F_s(\cdot) \sim F_s'(\cdot)$$

Subjective Probability Theory

- With state uniformity, $u_s(\cdot)$ and $u_{s'}(\cdot)$ can differ only up to an increasing linear transformation.
- That is, there is a utility function $u(\cdot)$ such that

$$u_s(\cdot) = \pi_s u(\cdot) + \beta_s$$
$$u_{s'}(\cdot) = \pi_{s'} u(\cdot) + \beta_{s'}$$

  for every state s and s', and for every $\pi_s, \pi_{s'} > 0$ and $\beta_s, \beta_{s'} > 0$.
- In words, the ranking between the expected utility of state s and s' remains unaffected.

Subjective Probability Theory

- ***Subjective probabilities EU theorem***:
  - Suppose that a preference relation satisfies continuity and the extended IA, and that preferences over lotteries are state uniform.
  - Then, there are subjective probabilities $(\pi_1, \pi_2, \dots, \pi_S) \gg 0$ and a utility function $u(\cdot)$ on certain amounts of money, such that for any two lists of monetary amounts $(x_1, x_2, \dots, x_S)$ and $(x_1', x_2', \dots, x_S')$,

$$(x_1, x_2, \dots, x_S) \succsim (x_1', x_2', \dots, x_S') \text{ iff}$$

$$\sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x_s')$$

Subjective Probability Theory

- *Intuition*: a decision maker prefers the first list of monetary outcomes to the second if the "subjective" expected utility from the first list is larger than that from the second.
- The predictions of the subjective EU theorem are not necessarily satisfied in all experimental settings.
  - Example: *Ellsberg paradox*

Subjective Probability Theory

- ***Ellsberg paradox***:
  - An urn contains 300 balls: 100 are red and the remaining 200 are either blue or green.
  - We first present the following two gambles to a group of students, asking each of them to choose either gamble A or B.
    - *Gamble A*: $1000 if the ball is red
    - *Gamble B*: $1000 if the ball is blue
  - We next present the following two gambles to the same group of students, asking each of them to choose either gamble C or D.
    - Gamble C: $1000 if the ball is not red
    - Gamble D: $1000 if the ball is not blue

Subjective Probability Theory

- ***Ellsberg paradox*** (continued):
  - Common choices: people choose A to B, and C to D.
  - But these choices violate subjective EU theory!
  - We know that

$$p(\text{Red}) = 1 - p(\text{not Red})$$
$$p(\text{Blue}) = 1 - p(\text{not Blue})$$

  - If gamble A is preferred to B, then we must have

$$p(\text{Red})u(\$1000) > p(\text{Blue})u(\$1000) \Longrightarrow$$
$$p(\text{Red}) > p(\text{Blue})$$

  - And if gamble C is preferred to D, then we must have

$$p(\text{not Red})u(\$1000) > p(\text{not Blue})u(\$1000) \Longrightarrow$$
$$p(\text{not Red}) > p(\text{not Blue})$$

  - But the above two expressions are incompatible.

Appendix 5.3: Ambiguity and Ambiguity Aversion

Ambiguity and Ambiguity Aversion

- Alternative theories that account for the anomaly in the Ellsberg paradox:
  1) expected utility theory with multiple priors (also referred to as *maxmin* expected utility)
  2) rank-dependent expected utility (or *Choquet* expected utility)
- Individuals have ambiguous (unclear) beliefs, rather than objective or subjective beliefs.
- Let f denote an act $f: s \rightarrow x$ from the set of states to the set of outcomes.

Ambiguity and Ambiguity Aversion

- ***Maxmin expected utility*** (MEU):
  - If subjects have too little information to form their priors, one could alternatively allow them to consider a set of priors.
  - If an individual is uncertainty averse, he will choose lottery f over another lottery g if the former provides a higher expected utility than the latter according to his worst possible prior.

Ambiguity and Ambiguity Aversion

- ***Uncertainty aversion***: Consider an individual who is indifferent between two lotteries f and g. Then, he is *uncertainty averse* if he weakly prefers the compound lottery $\alpha f + (1 - \alpha)g$ to lottery f, where $\alpha \in (0,1)$.
  - *Intuition*: a decision maker who is uncertainty averse has a preference for mixing (or hedging), since the compound lottery becomes at least as valuable as either of the two lotteries alone.

Ambiguity and Ambiguity Aversion

- ***Certainty-independence***: For any two lotteries f and g and a constant act k (i.e., a certain outcome or a lottery that remains constant across all states), the decision maker weakly prefers lottery f to g if and only if he prefers $\alpha f + (1-\alpha)k$ to $\alpha g + (1-\alpha)k$, where $\alpha \in (0,1)$.
  - Certainty-independence axiom relaxes the IA as it only requires that preferences over two lotteries to be unaffected when each lottery is mixed with a certain outcome k.

Ambiguity and Ambiguity Aversion

- A decision maker weakly prefers lottery f to g if and only if

$$\min_{p \in C} \int_S u(f(s))dp(s) \geq \min_{p \in C} \int_S u(g(s))dp(s)$$

- That is, the individual evaluates the expected utility of lotteries f and g according to each of his multiple priors $p \in C$, and then selects the lottery that yields the highest of the worst possible expected utilities.

Ambiguity and Ambiguity Aversion

- ***Example***:
  - Consider a decision maker with Bernoulli utility function $u(x) = \sqrt{x}$, where $x \geq 0$ denotes monetary amounts.
  - Assume that the decision maker faces two lotteries
$$L_A = (\$1, \$100)$$
$$L_B = (\$3, \$5)$$
  - Also, assume that the decision maker's priors are
$$(p_A, 1 - p_A) \text{ for } L_A$$
$$(p_B, 1 - p_B) \text{ for } L_B$$

Ambiguity and Ambiguity Aversion

- ***Example*** (continued):
  - According to MEU, the decision maker chooses lottery L_B if
$$\min_{p_B}[p_B\sqrt{3} + (1 - p_B)\sqrt{5}]$$
$$\geq \min_{p_A}[p_A\sqrt{1} + (1 - p_A)\sqrt{100}]$$
  - If the decision maker does not have any available information with which to update his priors, priors can take values $(p_A, p_B) \in (0,1)$.
  - It is possible that in his most pessimistic belief, he receives the lowest monetary amount with probability one.

Ambiguity and Ambiguity Aversion

- ***Example*** (continued):
  - Then, with argmin $p_B = 1$,
  $$\min_{p_B}[p_B\sqrt{3} + (1 - p_B)\sqrt{5}] = \sqrt{3}$$
  - Similarly, with argmin $p_A = 1$,
  $$\min_{p_A}[p_A\sqrt{1} + (1 - p_A)\sqrt{100}] = \sqrt{1}$$
  - Hence a decision maker with MEU preferences selects lotter L_B because $\sqrt{3} \geq \sqrt{1}$.

Ambiguity and Ambiguity Aversion

- ***Choquet expected utility*** (CEU):
  - Define beliefs with the use of capacities.
  - A capacity is defined as a real-valued function $v(\cdot)$ from a subset of the state space S to $[0,1]$, with the normalization $v(\emptyset) = 0$ and $v(S) = 1$.
  - If the capacity $v(\cdot)$ satisfies monotonicity, $v(A) \geq v(B)$, where A is a superset of B.
  - We cannot use a standard integral over states since the capacity $v(\cdot)$ does not correspond to our notion of beliefs.

Ambiguity and Ambiguity Aversion

– A decision maker weakly prefers f to g if the *Choquet integrals* satisfy

$$\int_S u\big(f(S)\big)dv(S) \geq \int_S u(g(S))dv(S)$$

– The CEU and MEU models are connected if we impose the uncertainty aversion axiom in CEU context. For that we need that capacity $v(\cdot)$ satisfies *supermodularity*, i.e.,

$$v(A \cup B) - v(B) \geq v(A \cup C) - v(C)$$

where C is a subset of B, i.e., $C \subset B$.

Ambiguity and Ambiguity Aversion

- ***Example***:
  - While the use of Choquet integrals is involved, the literature often uses "simple" capacities.
  - A simple capacity on state space S can be understood as a convex combination between two extreme capacities:
    1. a standard probability weight on A, $p(A) \in [0,1]$.
    2. the "complete ignorance" capacity w, where $w(S) = 1$ and $w(A) = 0$ for every $A \subseteq S$.

Ambiguity and Ambiguity Aversion

- ***Example*** (continued):
  - Formally, simple capacities are defined as
  $$v(A) = \lambda p(A) + (1 - \lambda) w(A)$$
  for every $A \subseteq S$ and where $\lambda \in [0,1]$.
  - Parameter λ denotes the individual's degree of confidence on $p(A)$, while $(1 - \lambda)$ captures his degree of ambiguity about $p(A)$.
  - For further reading, see Haller (2000) and Aflaki (2013).

Ambiguity and Ambiguity Aversion

- Further reading:

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