Advanced Microeconomic Theory

Chapter 4: Production Theory

Outline

- Production sets and production functions
- Profit maximization and cost minimization
- Cost functions
- Aggregate supply
- Efficiency (1st and 2nd FTWE)

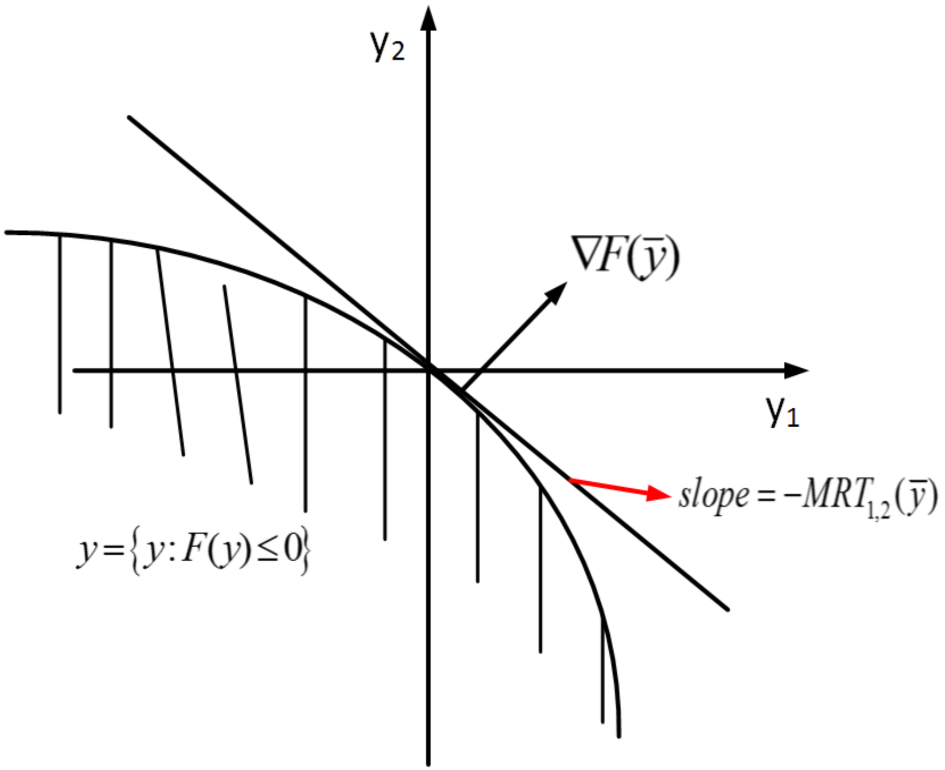
Production Sets and Production Functions

Production Sets

- Let us define a production vector (or plan)
$$y = (y_1, y_2, \dots, y_L) \in \mathbb{R}^L$$
  - If, for instance, $y_2 > 0$, then the firm is producing positive units of good 2 (i.e., good 2 is an *output*).
  - If, instead, $y_2 < 0$, then the firms is producing negative units of good 2 (i.e., good 2 is an *input*).
- Production plans that are technologically feasible are represented in the production set Y.
$$Y = \{y \in \mathbb{R}^L \colon F(y) \le 0\}$$
where $F(y)$ is the ***transformation function***.

Production Sets

- $F(y)$ can also be understood as a ***production function***.
- Firm uses units of y_1 as an input in order to produce units of y_2 as an output.
- Boundary of the production function is any production plan y such that $F(y) = 0$.
  - Also referred to as the ***transformation frontier***.



Production Sets

- For any production plan $\bar{y}$ on the production frontier, such that $F(\bar{y}) = 0$, we can totally differentiate $F(\bar{y})$ as follows

$$\frac{\partial F(\bar{y})}{\partial y_k} dy_k + \frac{\partial F(\bar{y})}{\partial y_l} dy_l = 0$$

  solving

$$\frac{dy_l}{dy_k} = -\frac{\frac{\partial F(\bar{y})}{\partial y_k}}{\frac{\partial F(\bar{y})}{\partial y_l}}, \text{ where } \frac{\frac{\partial F(\bar{y})}{\partial y_k}}{\frac{\partial F(\bar{y})}{\partial y_l}} = MRT_{l,k}(\bar{y})$$

  – $MRT_{l,k}(\bar{y})$ measures how much the (net) output k can increase if the firm decreases the (net) output of good l by one marginal unit.

Production Sets

- What if we denote input and outputs with different letters?

$$q = (q_1, q_2, \dots, q_M) \geq 0 \text{ outputs}$$
$$z = (z_1, z_2, \dots, z_{L-M}) \geq 0 \text{ inputs}$$

  where $L \geq M$.

- In this case, inputs are transformed into outputs by the production function, $f(z_1, z_2, \dots, z_{L-M})$, i.e., $f: \mathbb{R}^{L-M} \to \mathbb{R}^M$.

Production Sets

- ***Example***: When $M = 1$ (one single output), the production set Y can be described as
$$Y = \left\{ \begin{matrix} (-z_1, -z_2, \dots, -z_{L-1}, q): \\ q \leq f(z_1, z_2, \dots, z_{L-1}) \end{matrix} \right\}$$
- Holding the output level fixed, $dq = 0$, totally differentiate production function
$$\frac{\partial f(\bar{z})}{\partial z_k} dz_k + \frac{\partial f(\bar{z})}{\partial z_l} dz_l = 0$$

Production Sets

- ***Example*** (continued): and rearranging

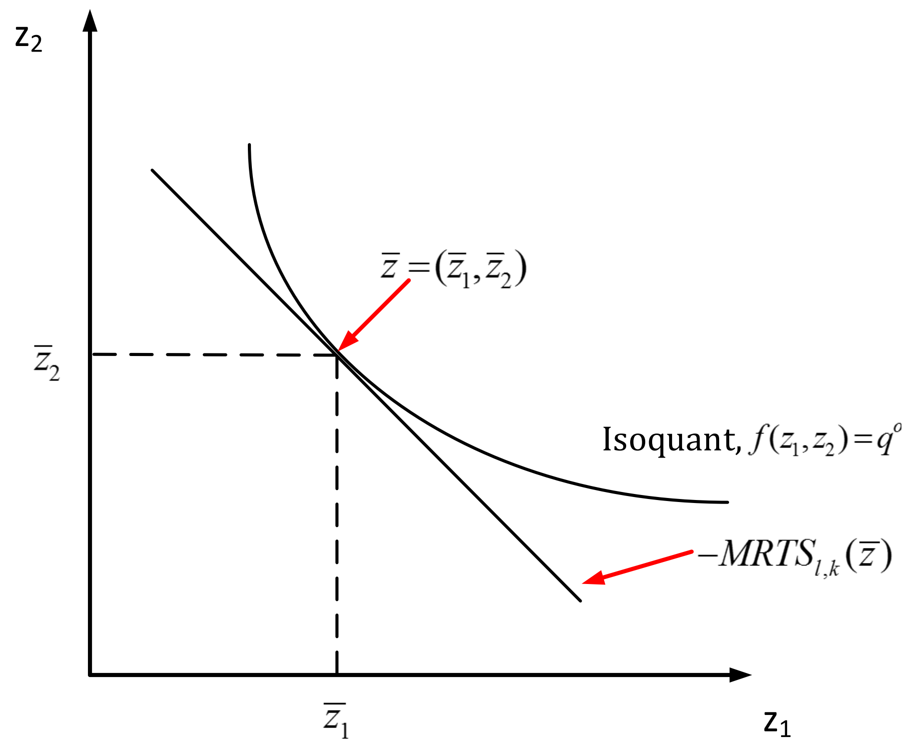
$$\frac{dz_l}{dz_k} = -\frac{\frac{\partial f(\bar{z})}{\partial z_k}}{\frac{\partial f(\bar{z})}{\partial z}}, \text{ where } \frac{\frac{\partial f(\bar{z})}{\partial z_k}}{\frac{\partial f(\bar{z})}{\partial z}} = MRTS_{l,k}(\bar{z})$$

- $MRTS_{l,k}(\bar{z})$ measures the additional amount of input k that must be used when we marginally decrease the amount of input l, and we want to keep output level at $\bar{q} = f(\bar{z})$.

- $MRTS_{l,k}(\bar{z})$ in production theory is analogous to the MRS in consumer theory, where we keep utility constant, $du = 0$.

Production Sets

- Combinations of (z_1, z_2) that produce the same total output q^0, i.e., $\{(z_1, z_2): f(z_1, z_2) = q^0\}$ is called ***isoquant***.
- The slope of the isoquant at $(\bar{z}_1, \bar{z}_2)$ is $MRTS_{l,k}(\bar{z})$.
- Remember:
  - $MRTS_{l,k}$ refers to isoquants (and production function).
  - $MRT_{l,k}$ refers to the transformation function.



Production Sets

- ***Example***: Find the $MRTS_{l,k}(z)$ for the Cobb-Douglas production function $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$, where $\alpha, \beta > 0$.
- The marginal product of input 1 is

$$\frac{\partial f(z_1, z_2)}{\partial z_1} = \alpha z_1^{\alpha-1} z_2^{\beta}$$

and that of input 2 is

$$\frac{\partial f(z_1, z_2)}{\partial z_1} = \beta z_1^{\alpha} z_2^{\beta-1}$$

Production Sets

- ***Example*** (continued): Hence, the $MRTS_{l,k}(z)$ is

$$MRTS_{l,k}(z) = \frac{\alpha z_1^{\alpha-1} z_2^{\beta}}{\beta z_1^{\alpha} z_2^{\beta-1}} = \frac{\alpha z_2^{\beta-(\beta-1)}}{\beta z_1^{\alpha-(\alpha-1)}} = \frac{\alpha z_2}{\beta z_1}$$

- For instance, for a particular vector $\bar{z} = (\bar{z}_1, \bar{z}_2) = (2,3)$, and $\alpha = \beta = \frac{1}{2}$, then

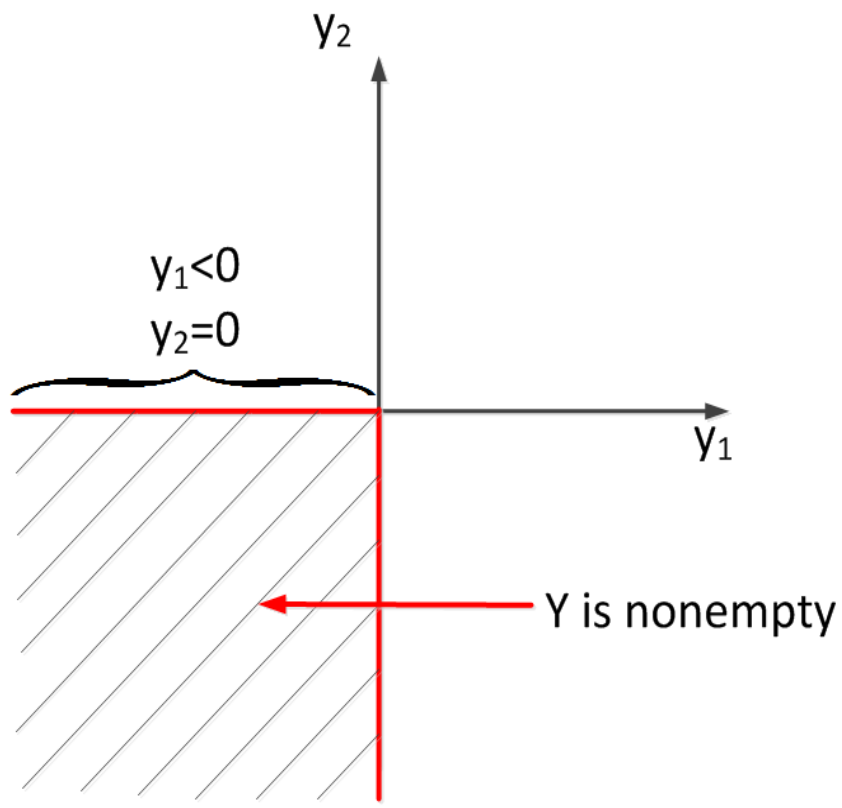
$$MRTS_{l,k}(\bar{z}) = \frac{3}{2} = 1.5$$

i.e., the slope of the isoquant evaluated at input vector $\bar{z} = (\bar{z}_1, \bar{z}_2) = (2,3)$ is -1.5.

Properties of Production Sets

1) ***Y is nonempty***: We have inputs and/or outputs.

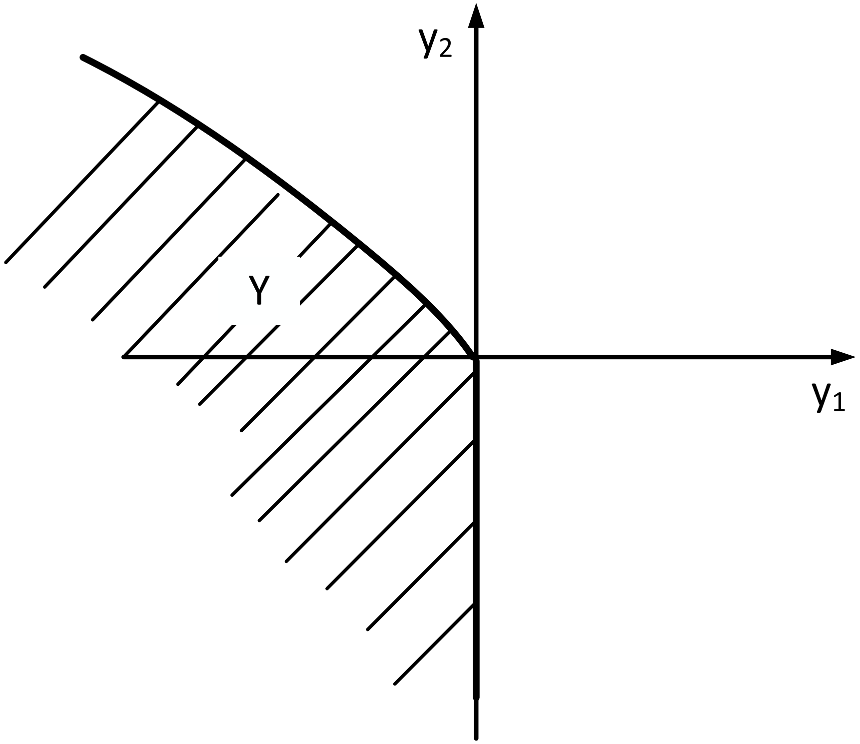
2) ***Y is closed***: The production set Y includes its boundary points.



Properties of Production Sets

3) No free lunch: No production with no resources.

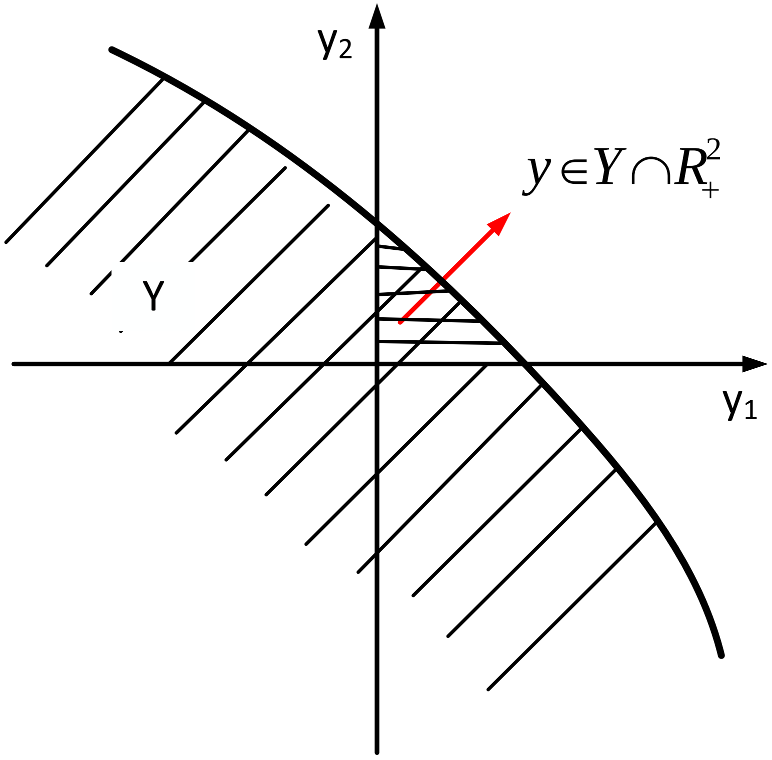
- The firm uses amounts of input y_1 in order to produce positive amounts of output y_2.



Properties of Production Sets

3) No free lunch: violation

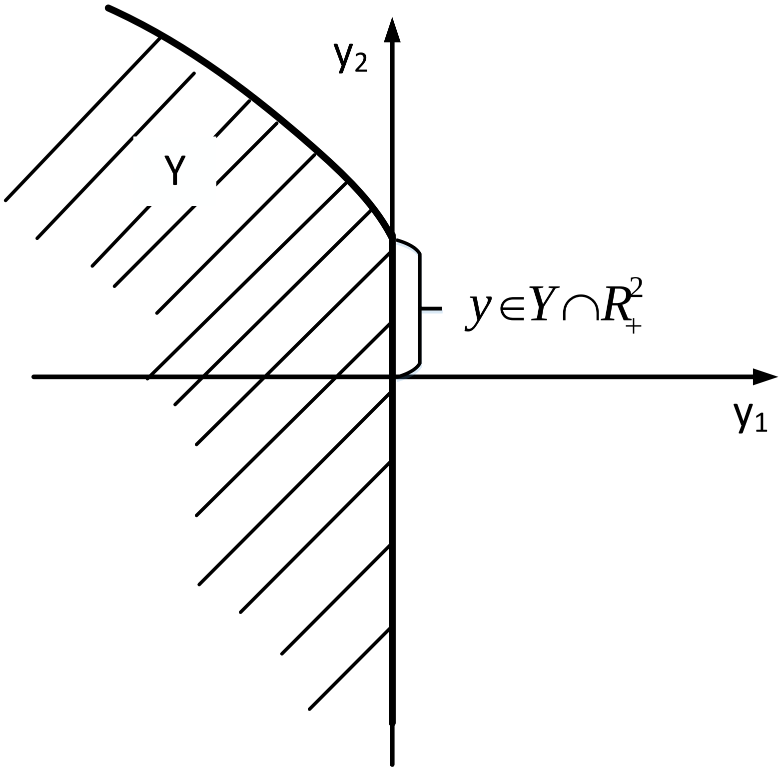
- The firm produces positive amounts of good 1 and 2 ($y_1 > 0$ and $y_2 > 0$) without the use of any inputs.



Properties of Production Sets

3) No free lunch: violation

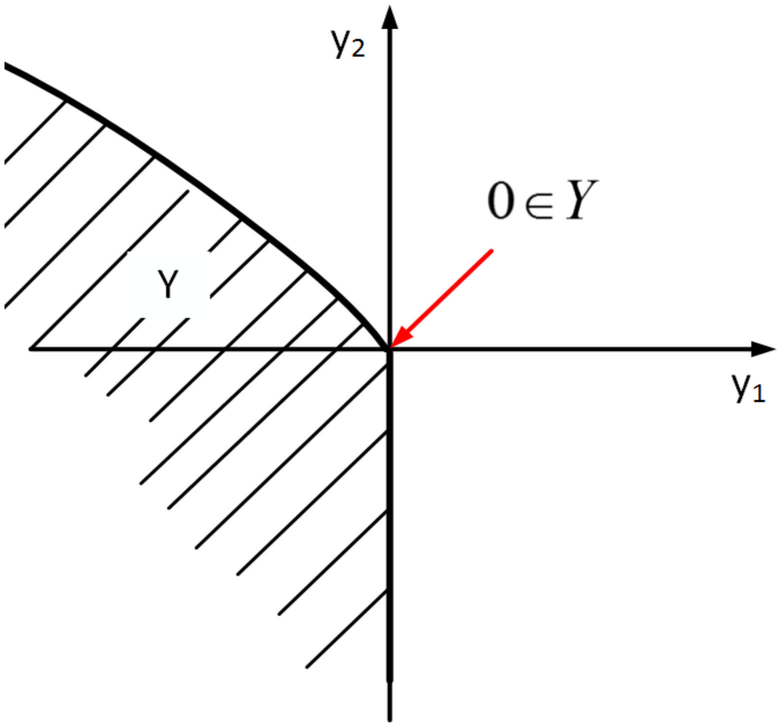
- The firm produces positive amounts of good 2 ($y_2 > 0$) with zero inputs, i.e., $y_1 = 0$.



Properties of Production Sets

4) Possibility of inaction

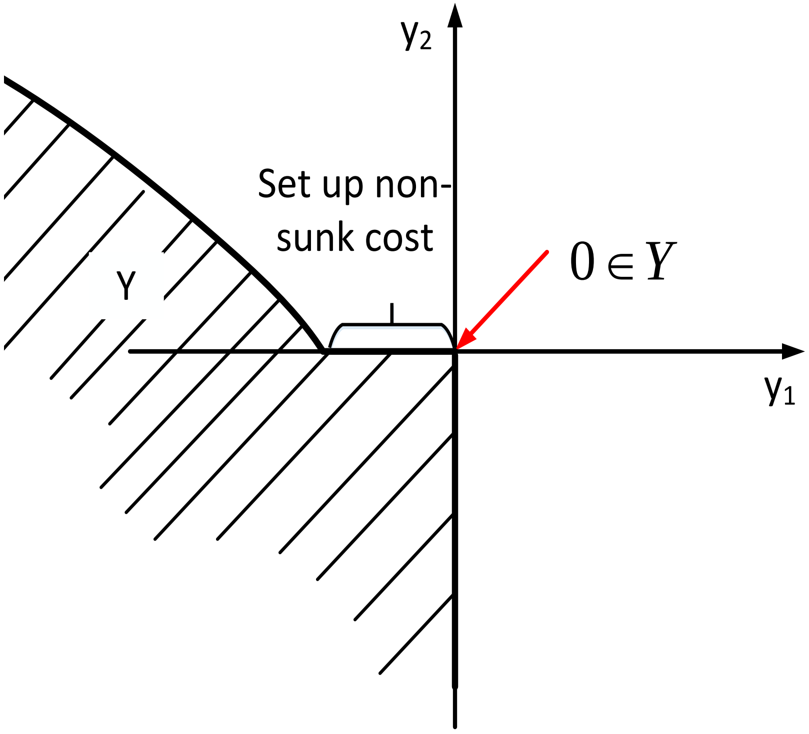
- Firm can choose to be inactive, using no inputs, and obtaining no output as a result (i.e., $0 \in Y$).



Properties of Production Sets

4) Possibility of inaction

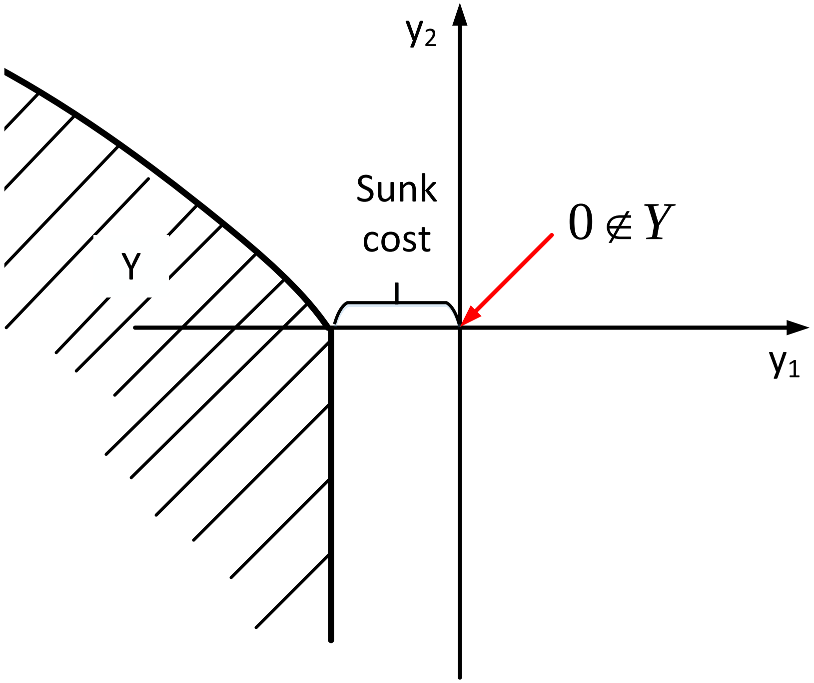
- Inaction is still possible when firms face *fixed costs* (i.e., $0 \in Y$).



Properties of Production Sets

4) Possibility of inaction

- Inaction is NOT possible when firms face *sunk costs* (i.e., $0 \notin Y$).



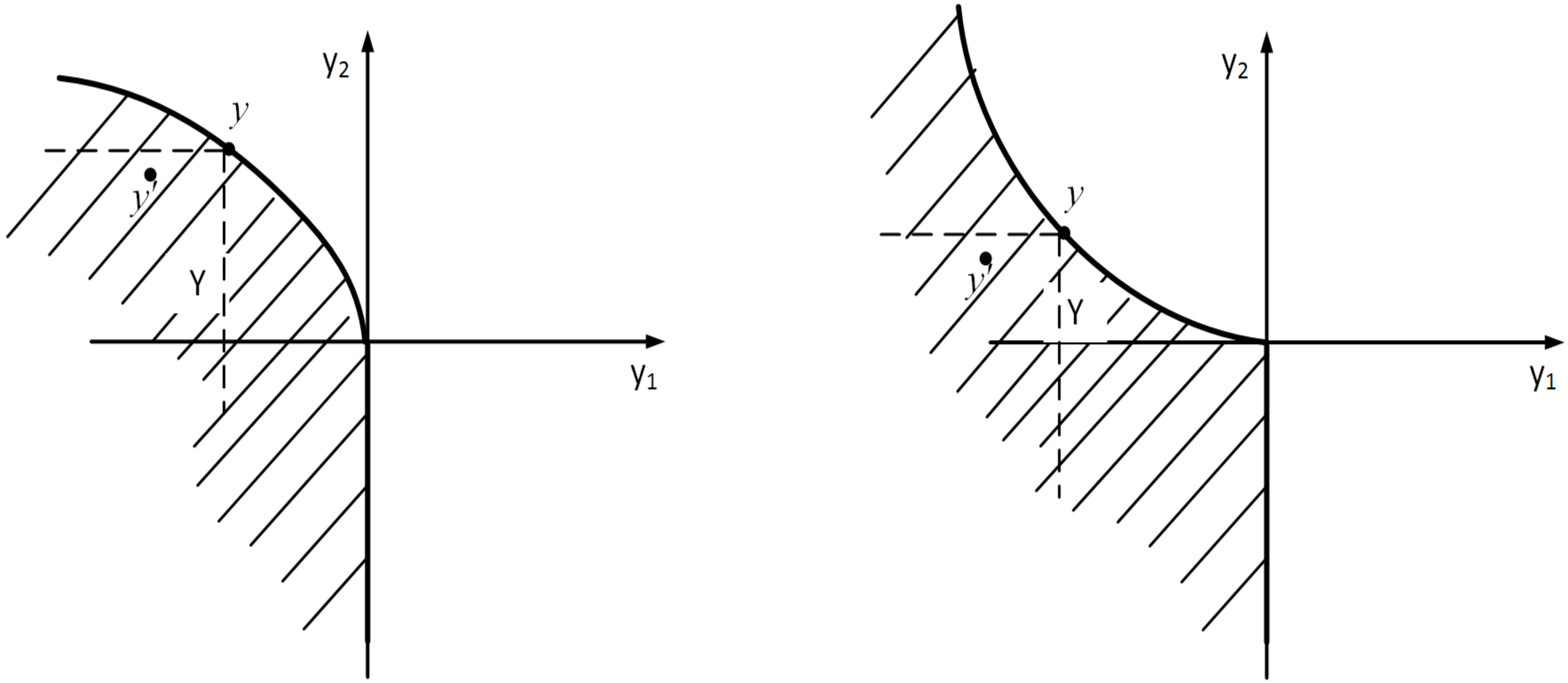
Properties of Production Sets

5) Free disposal: if $y \in Y$ and $y' \leq y$, then $y' \in Y$.

- y' is less efficient than y:
  - Either it produces the same amount of output with more inputs, or less output using the same inputs.
- Then, y' also belongs to the firm's production set.
- That is, the producer can use more inputs without the need to reduce his output:
  - The producer can dispose of (or eliminate) this additional inputs at no cost.

Properties of Production Sets

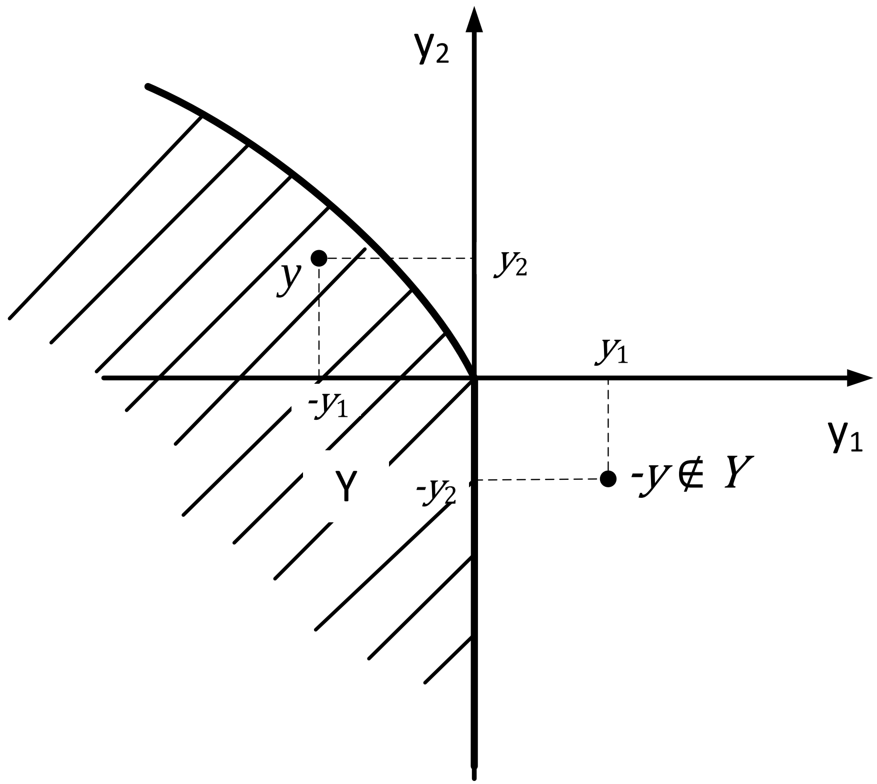
5) Free disposal (continued)



Properties of Production Sets

6) Irreversibility

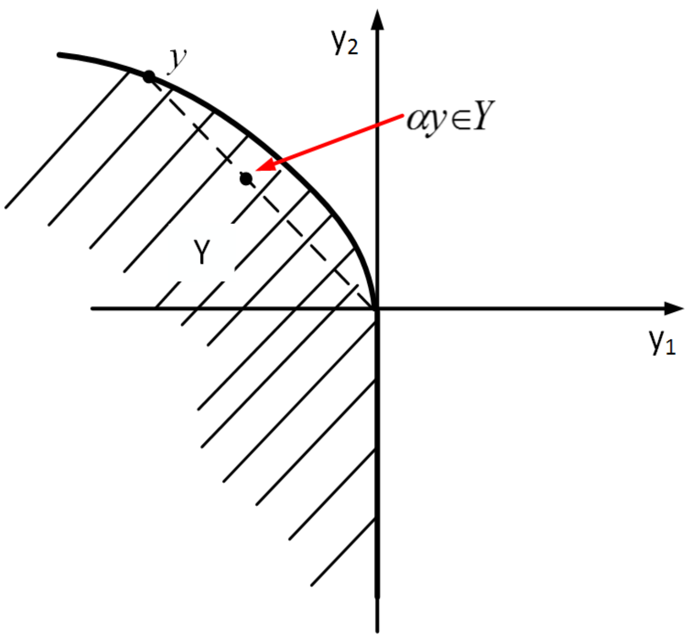
- Suppose that $y \in Y$ and $y \neq 0$. Then, $-y \notin Y$.
- "No way back"

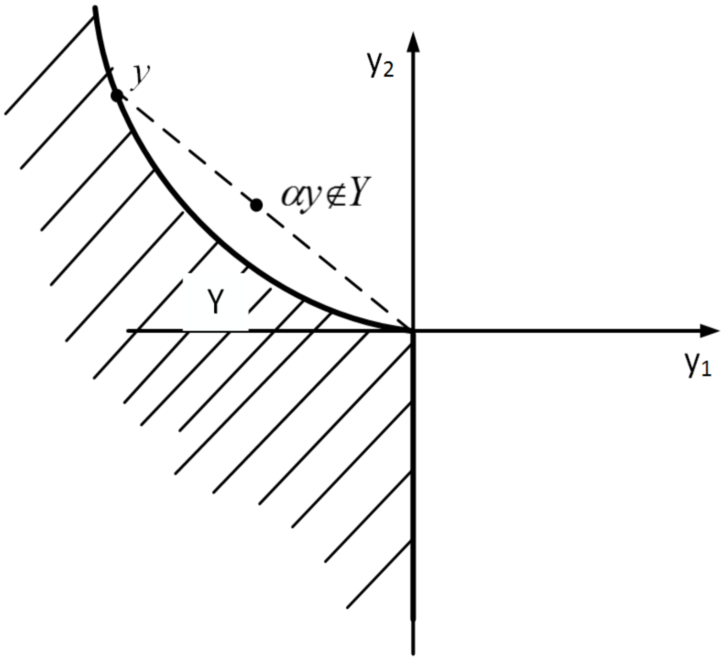


Properties of Production Sets

7) Non-increasing returns to scale: If $y \in Y$, then $\alpha y \in Y$ for any $\alpha \in [0,1]$.

- That is, any feasible vector can be scaled down.

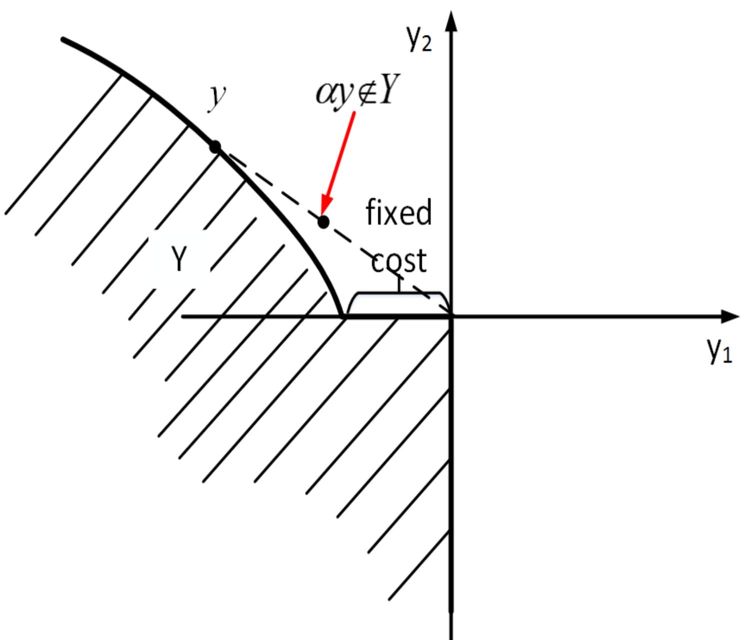


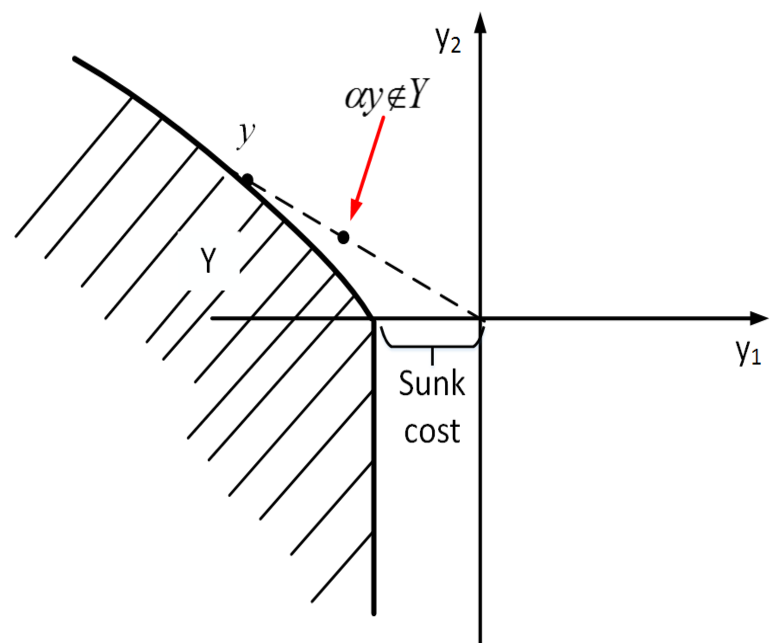


Properties of Production Sets

7) Non-increasing returns to scale

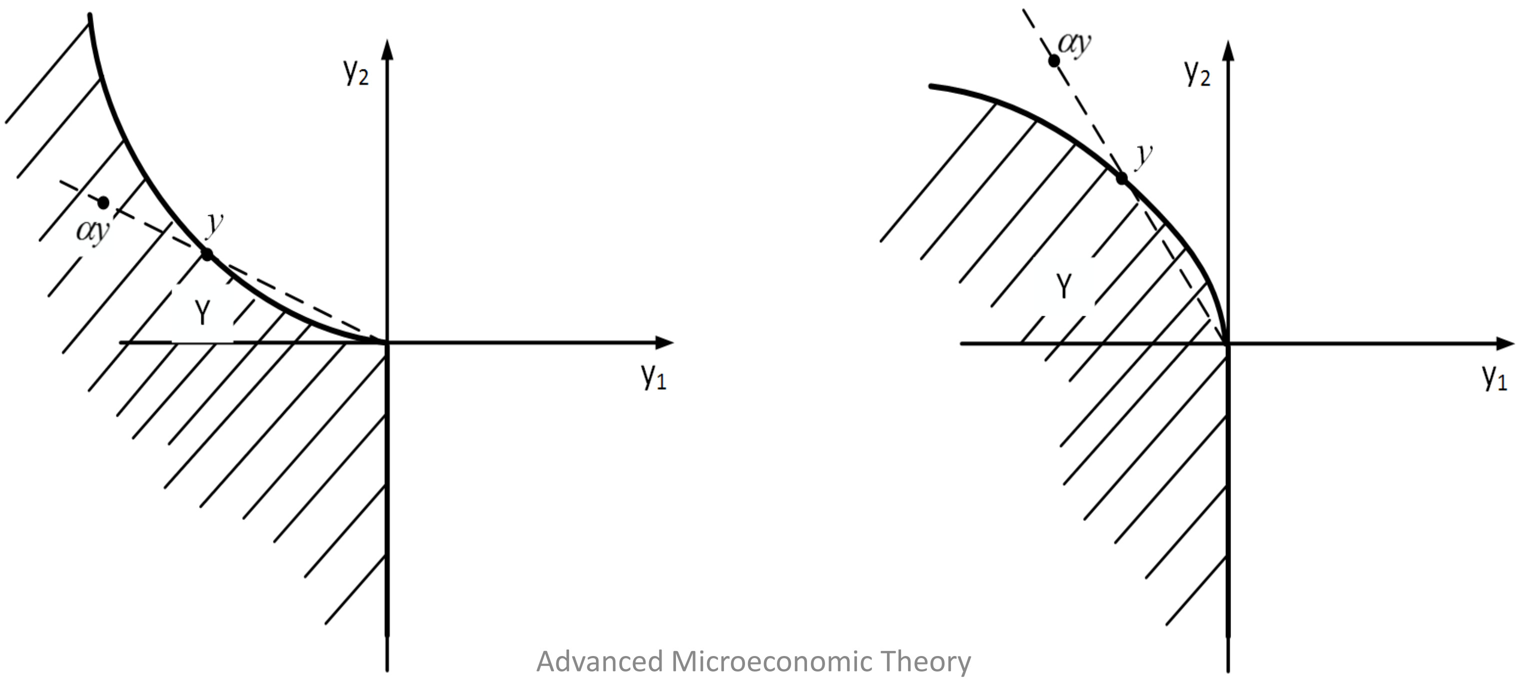
- The presence of fixed or sunk costs *violates* non-increasing returns to scale.





Properties of Production Sets

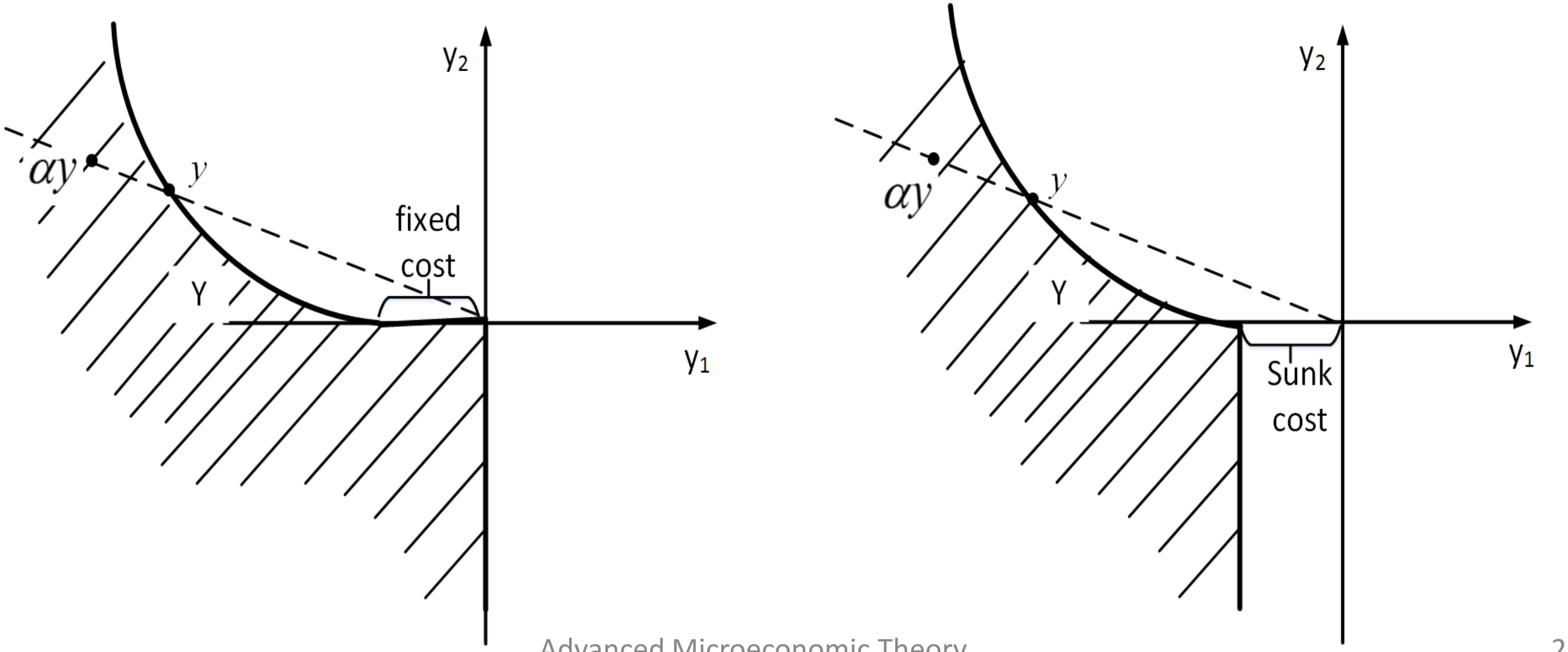
- ***8) Non-decreasing returns to scale***: If $y \in Y$, then $\alpha y \in Y$ for any $\alpha \geq 1$.
  - That is, any feasible vector can be scaled up.



Properties of Production Sets

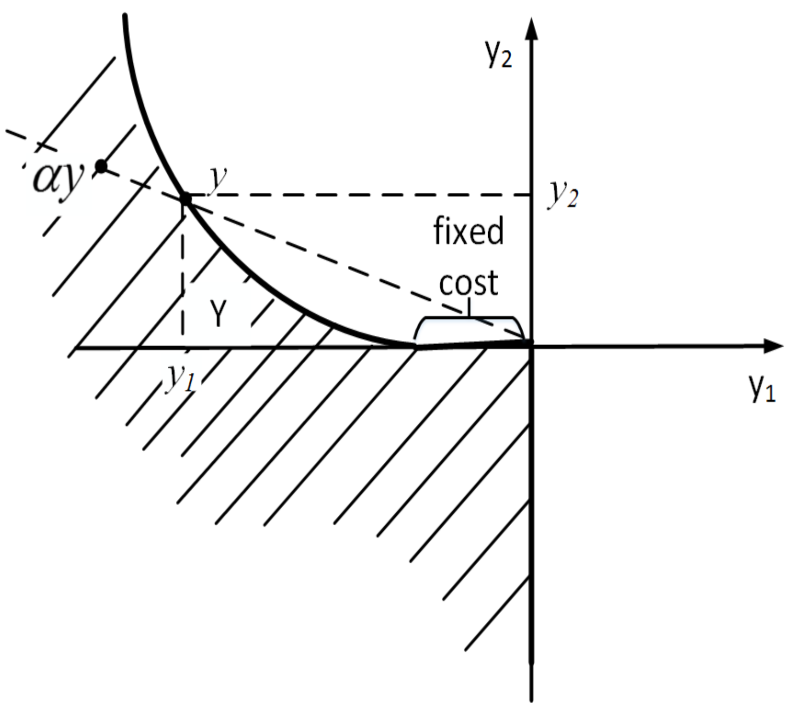
8) Non-decreasing returns to scale

- The presence of fixed or sunk costs do NOT violate non-decreasing returns to scale.



Properties of Production Sets

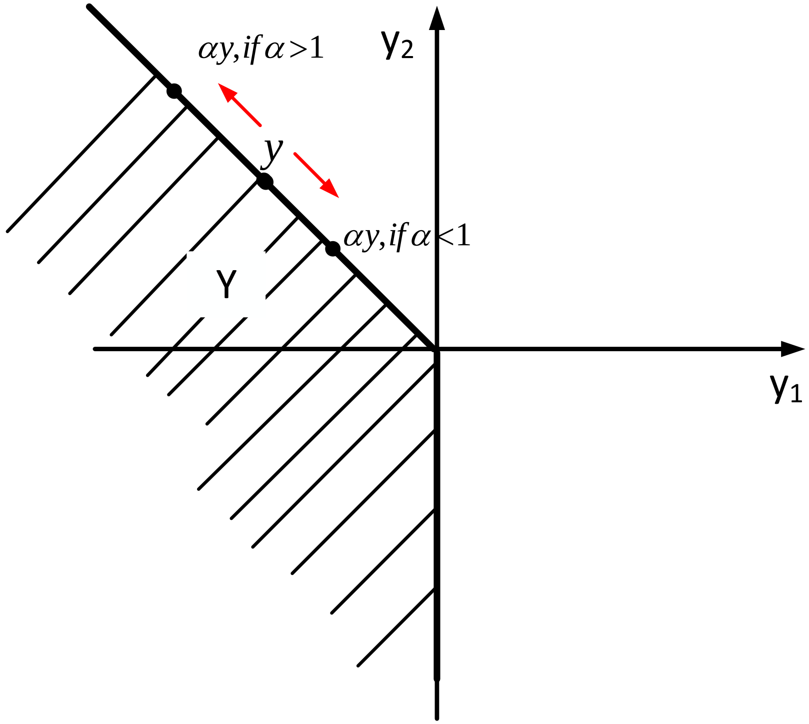
- ***Returns to scale***:
  - When scaling up/down a given production plan $y = (-y_1, y_2)$:
    - We connect y with a ray from the origin.
    - Then, the ratio $\frac{y_2}{y_1}$ must be maintained in all points along the ray.
    - Note that the angle of the ray is exactly this ratio $\frac{y_2}{y_1}$.



Properties of Production Sets

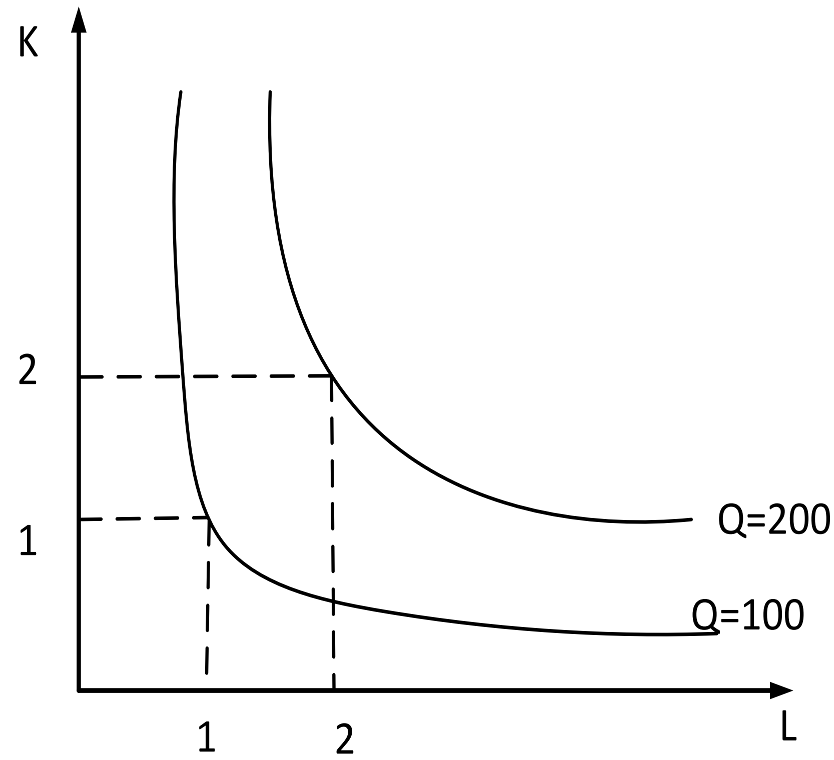
9) Constant returns to scale (CRS): If $y \in Y$, then $\alpha y \in Y$ for any $\alpha \geq 0$.

- CRS is non-increasing and non-decreasing.



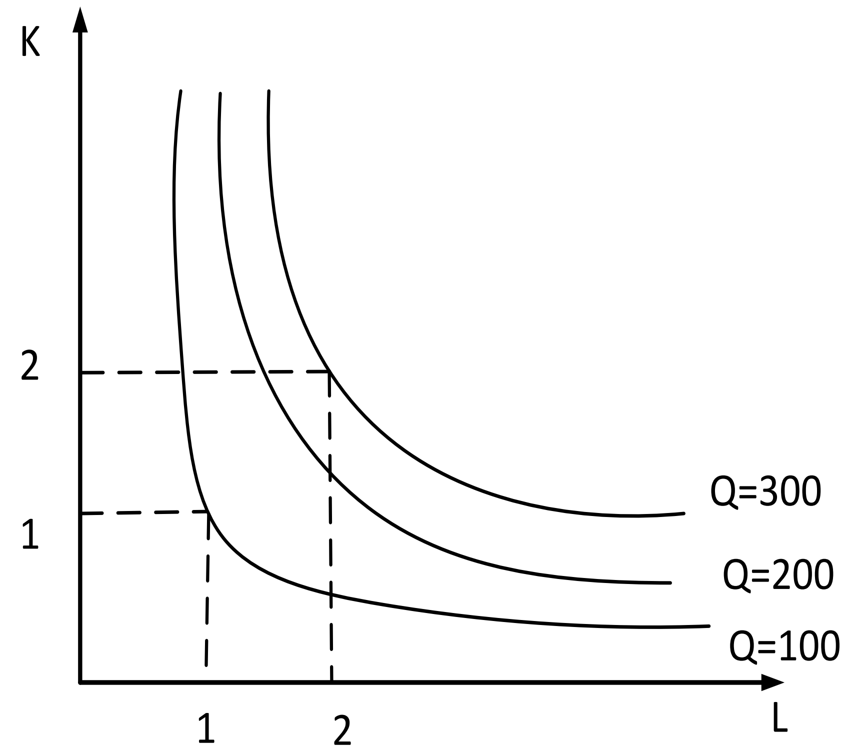
Properties of Production Sets

- Alternative graphical representation of *constant returns to scale*:
  - Doubling K and L doubles output (i.e., proportionally increase in output).



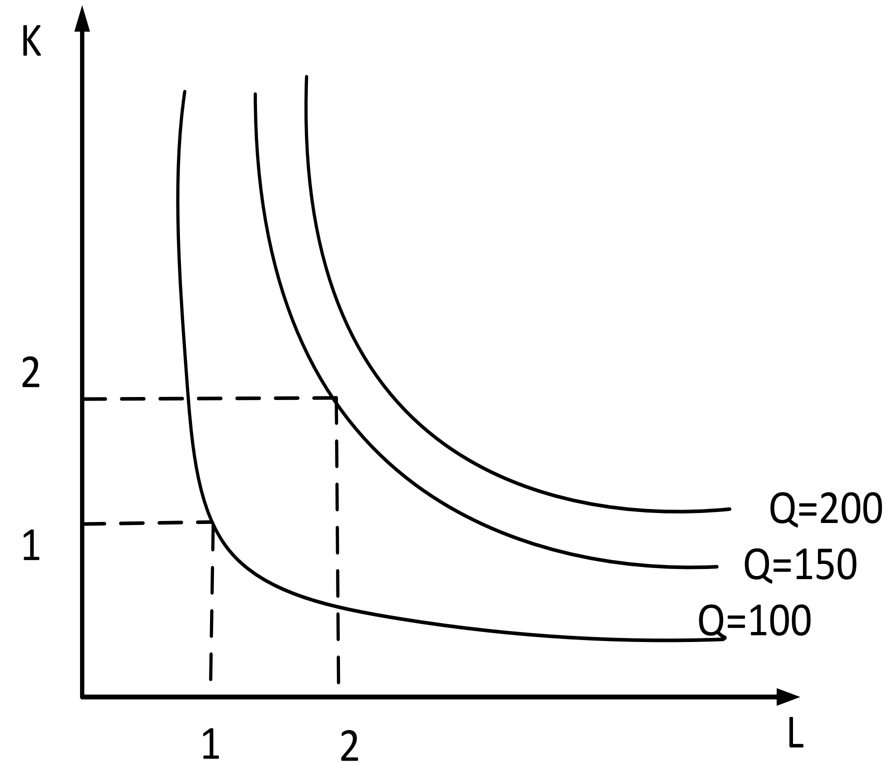
Properties of Production Sets

- Alternative graphical representation of *increasing-returns to scale*:
  - Doubling K and L increases output more than proportionally.



Properties of Production Sets

- Alternative graphical representation of *decreasing-returns to scale*:
  - Doubling K and L increases output less than proportionally.



Properties of Production Sets

- ***Example***: Let us check returns to scale in the Cobb-Douglas production function $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$. Increasing all arguments by a common factor λ, we obtain

$$f(z_1, z_2) = (\lambda z_1)^{\alpha} (\lambda z_2)^{\beta} = \lambda^{\alpha+\beta} z_1^{\alpha} z_2^{\beta}$$

  – When $\alpha + \beta = 1$, we have constant returns to scale;
  – When $\alpha + \beta > 1$, we have increasing returns to scale;
  – When $\alpha + \beta < 1$, we have decreasing returns to scale.

Properties of Production Sets

- Returns to scale in different US industries (Source: Hsieh, 1995):

	Industry	$\alpha + \beta$
Decreasing returns	Tobacco	0.51
	Food	0.91
Constant returns	Apparel and textile	1.01
	Furniture	1.02
	Electronics	1.02
Increasing returns	Paper products	1.09
	Petroleum and coal	1.18
	Primary metal	1.24

Properties of Production Sets

Homogeneity of the Production Function	Returns to Scale
$K = 1$	Constant Returns
$K > 1$	Increasing Returns
$K < 1$	Decreasing Returns

Properties of Production Sets

- The *linear* production function exhibits CRS as increasing all inputs by a common factor t yields

$$f(tk, tl) = atk + btl = t(ak + bl) \equiv tf(k, l)$$

- The *fixed proportion* production function $f(k, l) = \min\{ak, bl\}$ also exhibits CRS as

$$f(tk, tl) = \min\{atk, btl\} = t \cdot \min\{ak, bl\} \equiv tf(k, l)$$

Properties of Production Sets

- Increasing/decreasing returns to scale can be incorporated into a production function $f(k,l)$ exhibiting CRS by using a transformation function $F(\cdot)$

$$F(k,l) = [f(k,l)]^{\gamma}, \text{ where } \gamma > 0$$

- Indeed, increasing all arguments by a common factor t, yields

$$F(tk,tl) = [f(tk,tl)]^{\gamma} = \Bigg[\overbrace{t \cdot f(k,l)}^{\text{by CRS of } f(\cdot)}\Bigg]^{\gamma}$$

$$= t^{\gamma} \cdot \underbrace{[f(k,l)]^{\gamma}}_{F(k,l)} = t^{\gamma} \cdot F(k,l)$$

Properties of Production Sets

- Hence,
  - if $\gamma > 1$, the transformed production function $F(k, l)$ exhibits increasing returns to scale;
  - if $\gamma < 1$, the transformed production function $F(k, l)$ exhibits decreasing returns to scale;

Properties of Production Sets

- ***Scale elasticity***: an alternative measure of returns to scale.
  - It measures the percent increase in output due to a 1% increase in the amounts of all inputs

$$\varepsilon_{q,t} = \frac{\partial f(tk,tl)}{\partial t} \cdot \frac{t}{f(k,l)}$$

  where t denotes the common increase in all inputs.
  - *Practice*: Show that, if a function exhibits CRS, then it has a scale elasticity of $\varepsilon_{q,t}$=1.

Properties of Production Sets

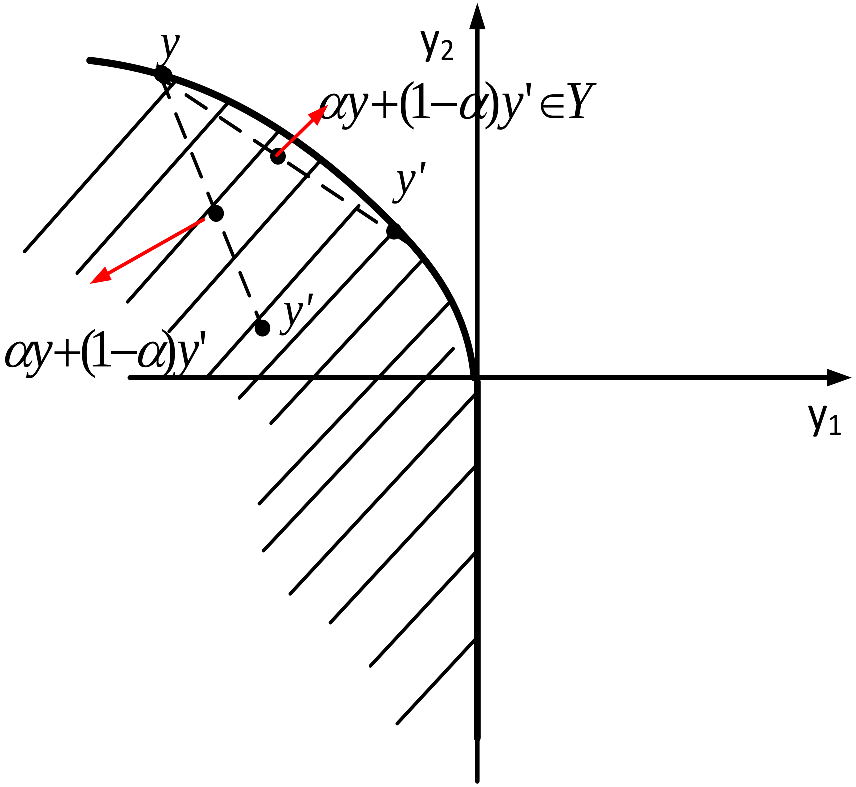
10) Additivity (or free entry): If $y \in Y$ and $y' \in Y$, then $y + y' \in Y$.

- *Interpretation*: one plant produces y, while another plant enters the market producing y'. Then, the aggregate production $y + y'$ is feasible.

Properties of Production Sets

11) Convexity: If $y, y' \in Y$ and $\alpha \in [0,1]$, then $\alpha y + (1-\alpha)y' \in Y$.

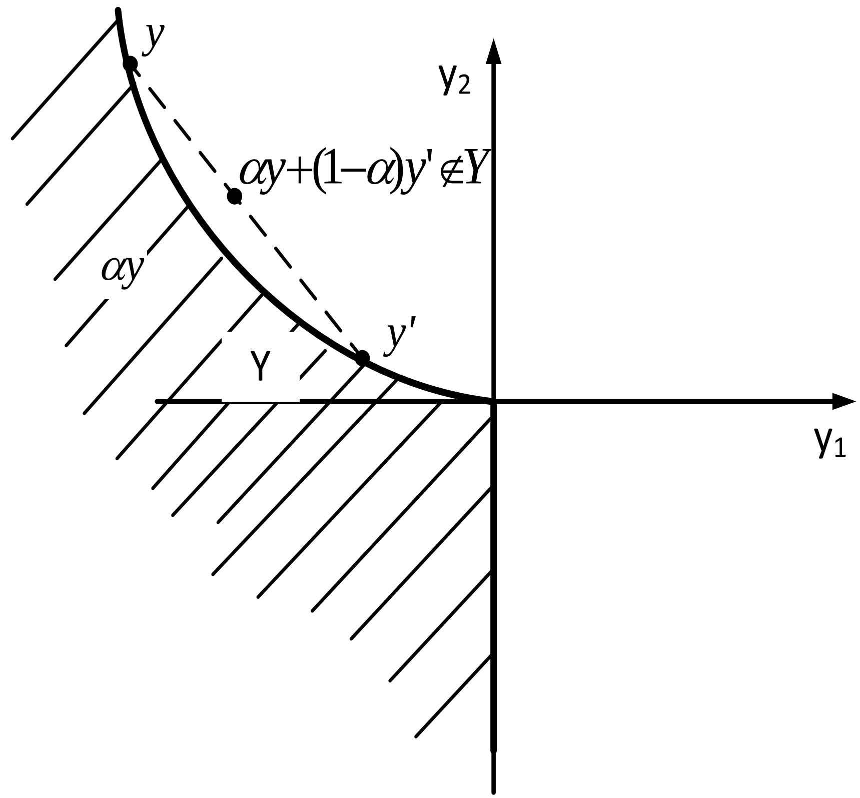
Intuition: "balanced" input-output combinations are more productive than "unbalanced" ones.



Properties of Production Sets

11) Convexity: violation

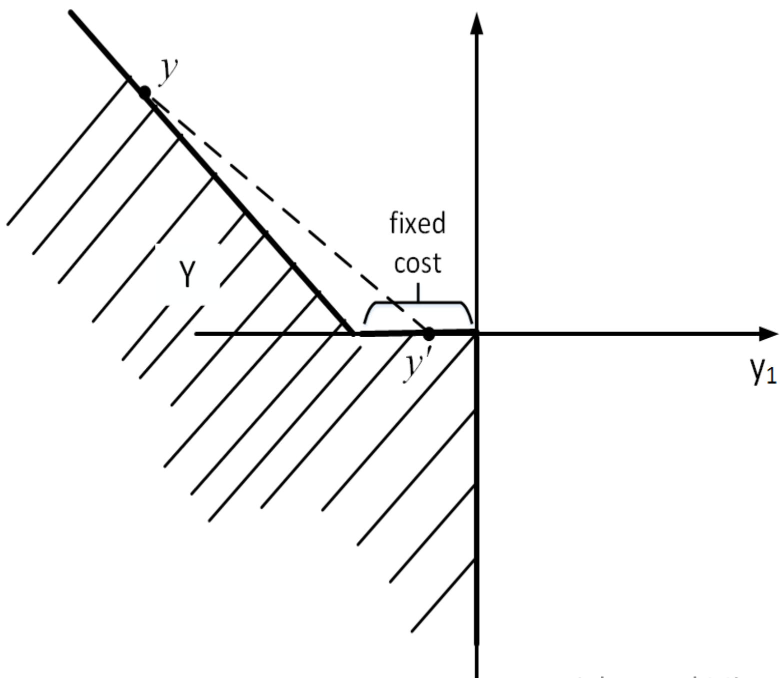
Note: The convexity of the production set maintains a close relationship with the concavity of the production function.

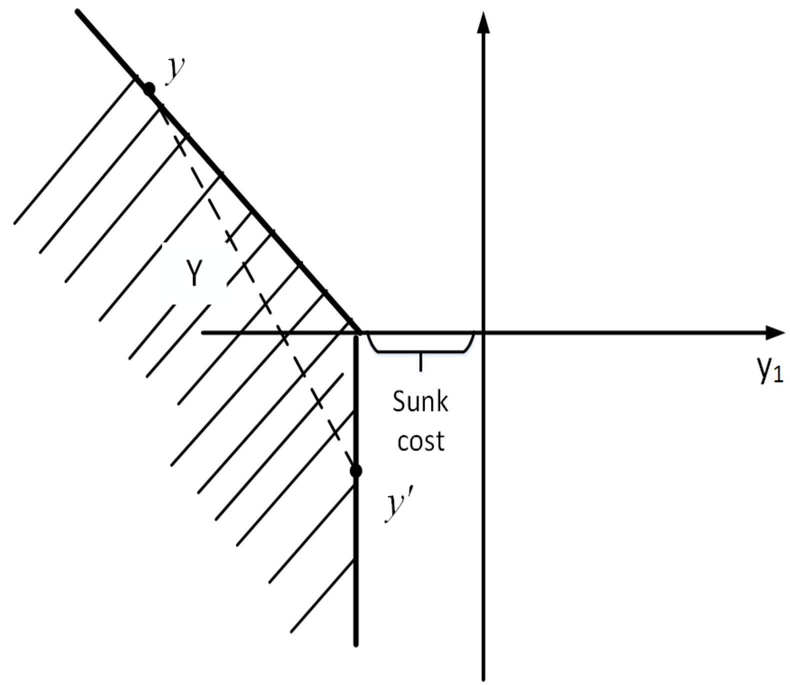


Properties of Production Sets

11) Convexity

- With fixed costs, convexity is NOT necessarily satisfied;
- With sunk costs, convexity is satisfied.





Diminishing MRTS

- The slope of the firm's isoquants is

$$MRTS_{l,k} = -\frac{dk}{dl}, \text{ where } MRTS_{l,k} = \frac{f_l}{f_k}$$

- Differentiating $MRTS_{l,k}$ with respect to labor yields

$$\frac{\partial MRTS_{l,k}}{\partial l} = \frac{f_k\left(f_{ll} + f_{lk} \cdot \frac{dk}{dl}\right) - f_l\left(f_{kl} + f_{kk} \cdot \frac{dk}{dl}\right)}{(f_k)^2}$$

Diminishing MRTS

- Using the fact that $\frac{dk}{dl} = -\frac{f_l}{f_k}$ along an isoquant and Young's theorem $f_{lk} = f_{kl}$,

$$\frac{\partial MRTS_{l,k}}{\partial l} = \frac{f_k\left(f_{ll} - f_{lk} \cdot \frac{f_l}{f_k}\right) - f_l\left(f_{kl} - f_{kk} \cdot \frac{f_l}{f_k}\right)}{(f_k)^2}$$

$$= \frac{f_k f_{ll} - f_{lk} f_l - f_l f_{kl} + f_{kk} \cdot \frac{f_l^2}{f_k}}{(f_k)^2}$$

Diminishing MRTS

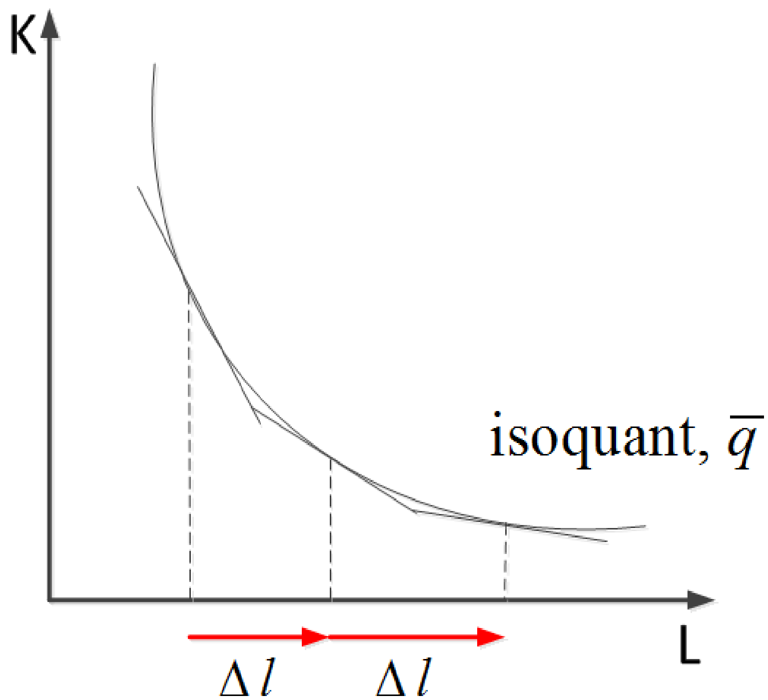
- Multiplying numerator and denominator by f_k

$$\frac{\partial MRTS_{l,k}}{\partial l} = \frac{\overbrace{f_k^2}^{+}\overbrace{f_{ll}}^{-} + \overbrace{f_{kk}}^{-}\overbrace{f_l^2}^{+} - \overbrace{2f_l f_k}^{+}\overbrace{f_{lk}}^{-\text{ or}+}}{(f_k)^3}$$

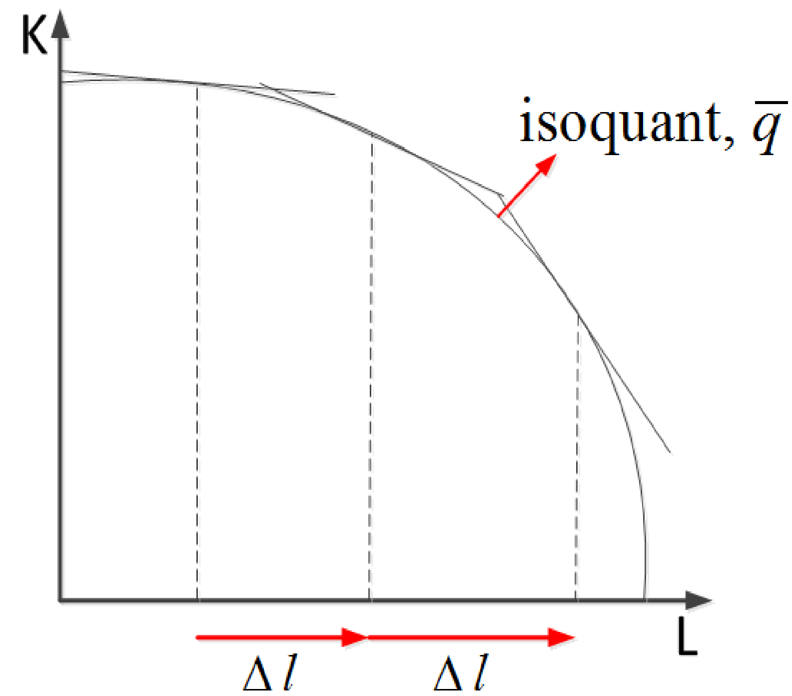
- Thus,
  - If $f_{lk} > 0$ (i.e., $\uparrow k \Longrightarrow \uparrow MP_l$), then $\frac{\partial MRTS_{l,k}}{\partial l} < 0$
  - If $f_{lk} < 0$, then we have

$$\left|f_k^2 f_{ll} + f_{kk} f_l^2\right| \begin{Bmatrix} > \\ < \end{Bmatrix} |2f_l f_k f_{lk}| \Longrightarrow \frac{\partial MRTS_{l,k}}{\partial l} \begin{Bmatrix} < \\ > \end{Bmatrix} 0$$

Diminishing MRTS



$f_{lk} > 0$ ($\uparrow k \Longrightarrow \uparrow MP_l$), or
$f_{lk} < 0$ ($\uparrow k \Longrightarrow \downarrow MP_l$) but
small $\downarrow$ in MP_l



$f_{lk} < 0$ ($\uparrow k \Longrightarrow \downarrow\downarrow MP_l$)

Diminishing MRTS

- ***Example***: Let us check if the production function $f(k,l) = 600k^2l^2 - k^3l^3$ yields convex isoquants.
  - Marginal products:

    $$MP_l = f_l = 1{,}200k^2l - 3k^3l^2 > 0 \text{ iff } kl < 400$$
    $$MP_k = f_k = 1{,}200kl^2 - 3k^2l^3 > 0 \text{ iff } kl < 400$$

  - Decreasing marginal productivity:

    $$\frac{\partial MP_l}{\partial l} = f_{ll} = 1{,}200k^2 - 6k^3l < 0 \text{ iff } kl > 200$$
    $$\frac{\partial MP_k}{\partial k} = f_{kk} = 1{,}200l^2 - 6kl^3 < 0 \text{ iff } kl > 200$$

Diminishing MRTS

- ***Example*** (continued):
  - Is $200 < kl < 400$ then sufficient condition for diminishing $MRTS_{l,k}$?
    - No!
    - We need $f_{kl} > 0$ too in order to guarantee diminishing $MRTS_{l,k}$.
  - Check the sign of f_{lk}:

$$f_{lk} = f_{kl} = 2{,}400kl - 9k^2l^2 > 0 \text{ iff } kl < 266$$

Diminishing MRTS

- ***Example*** (continued):
  - Alternatively, we can represent the above conditions by solving for l in the above inequalities:

$$MP_l > 0 \text{ iff } l < \frac{400}{k} \qquad \frac{\partial MP_l}{\partial l} < 0 \text{ iff } l > \frac{200}{k}$$

$$MP_k > 0 \text{ iff } l < \frac{400}{k} \qquad \frac{\partial MP_k}{\partial k} < 0 \text{ iff } l > \frac{200}{k}$$

and

$$f_{lk} > 0 \text{ iff } l < \frac{266}{k}$$

Diminishing MRTS

- ***Example*** (continued):
  - Hence, $\frac{200}{k} < l < \frac{266}{k}$ guarantees positive but diminishing marginal products and, in addition, a diminishing $MRTS_{l,k}$.
  - ***Figure***: $l = \frac{200}{k}$ is a curve decreasing in k, never crossing either axes. Similarly for $l = \frac{266}{k}$.

Constant Returns to Scale

- If production function $f(k,l)$ exhibits CRS, then increasing all inputs by a common factor t yields

$$f(k,l) = tf(k,l)$$

- Hence, $f(k,l)$ is homogenous of degree 1, thus implying that its first-order derivatives

$$f_k(k,l) \text{ and } f_l(k,l)$$

are homogenous of degree zero.

Constant Returns to Scale

- Therefore,

$$MP_l = \frac{\partial f(k,l)}{\partial l} = \frac{\partial f(tk,tl)}{\partial l}$$
$$= f_l(k,l) = f_l(tk,tl)$$

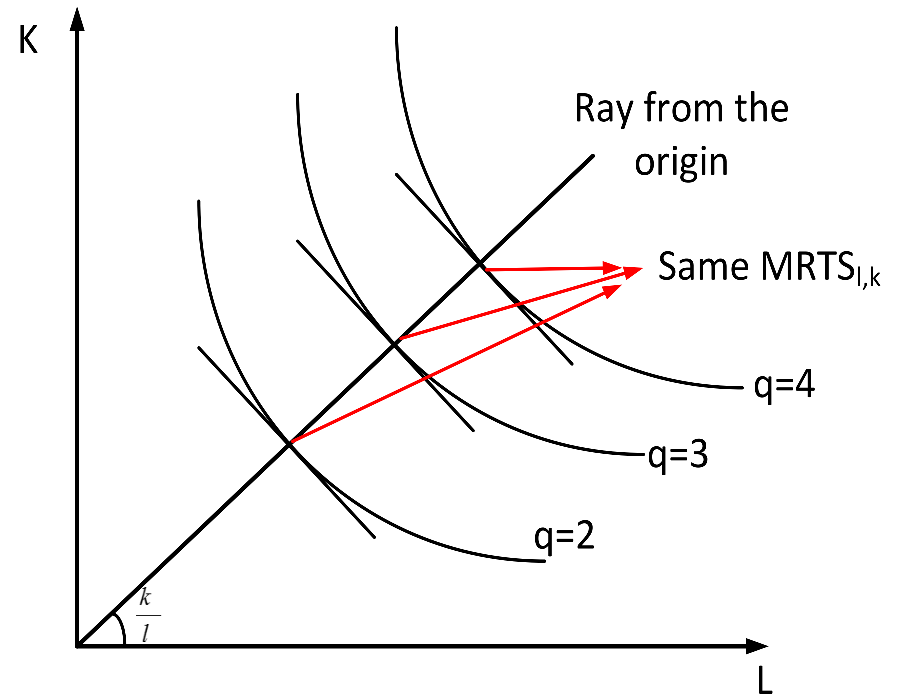
- Setting $t = \frac{1}{l}$, we obtain

$$MP_l = f_l(k,l) = f_l\left(\frac{1}{l}k, \frac{k}{k}\right) = f_l\left(\frac{k}{l}, 1\right)$$

- Hence, MP_l only depends on the ratio $\frac{k}{l}$, but not on the absolute levels of k and l that firm uses.
- A similar argument applies to MP_k.

Constant Returns to Scale

- Thus, $MRTS_{l,k} = \frac{MP_l}{MP_k}$ only depends on the ratio of capital to labor.
- The slope of a firm's isoquants coincides at any point along a ray from the origin.
- Firm's production function is, hence, homothetic.



Elasticity of Substitution

Elasticity of Substitution

- ***Elasticity of substitution (σ)*** measures the proportionate change in the k/l ratio relative to the proportionate change in the $MRTS_{l,k}$ along an isoquant:

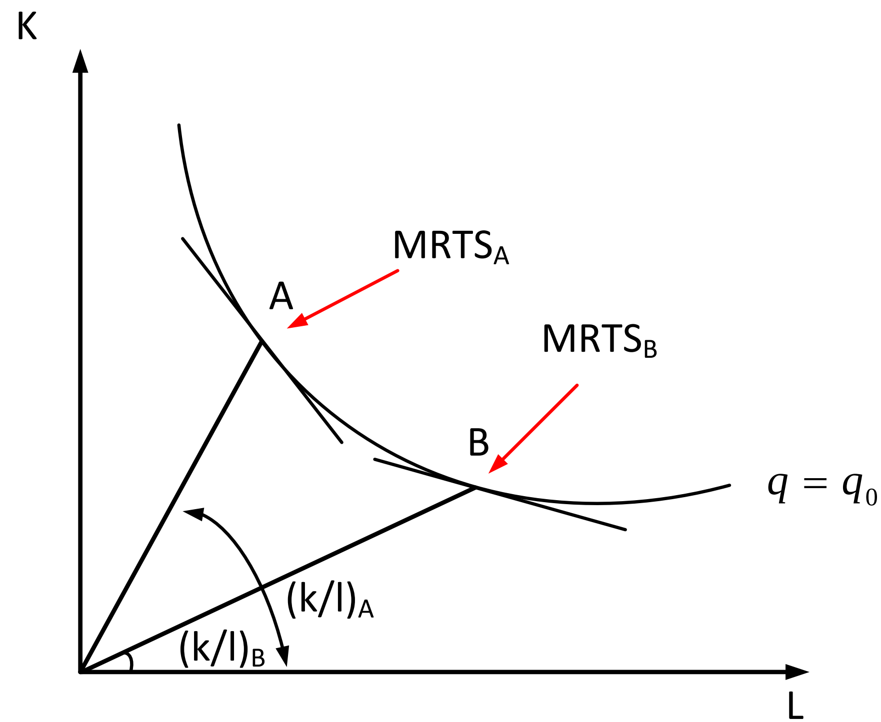
$$\sigma = \frac{\%\Delta(k/l)}{\%\Delta MRTS_{l,k}} = \frac{d(k/l)}{dMRTS_{l,k}} \cdot \frac{MRTS_{l,k}}{k/l}$$

$$= \frac{\partial \ln(k/l)}{\partial \ln(MRTS)_{l,k}}$$

where $\sigma > 0$ as k/l and $MRTS_{l,k}$ move in the same direction.

Elasticity of Substitution

- Both $MRTS_{l,k}$ and k/l will change as we move from point A to point B.
- σ is the ratio of these changes.
- σ measures the curvature of the isoquant.

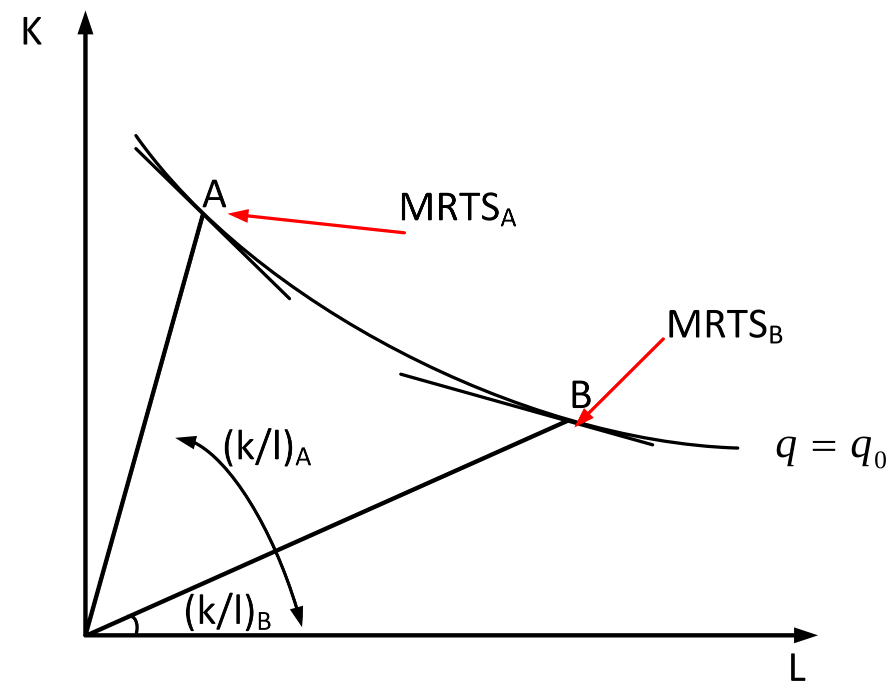


Elasticity of Substitution

- If we define the elasticity of substitution between two inputs to be proportionate change in the ratio of the two inputs to the proportionate change in $MRTS$, we need to hold:
  - output constant (so we move along the same isoquant), and
  - the levels of other inputs constant (in case we have more than two inputs). For instance, we fix the amount of other inputs, such as land.

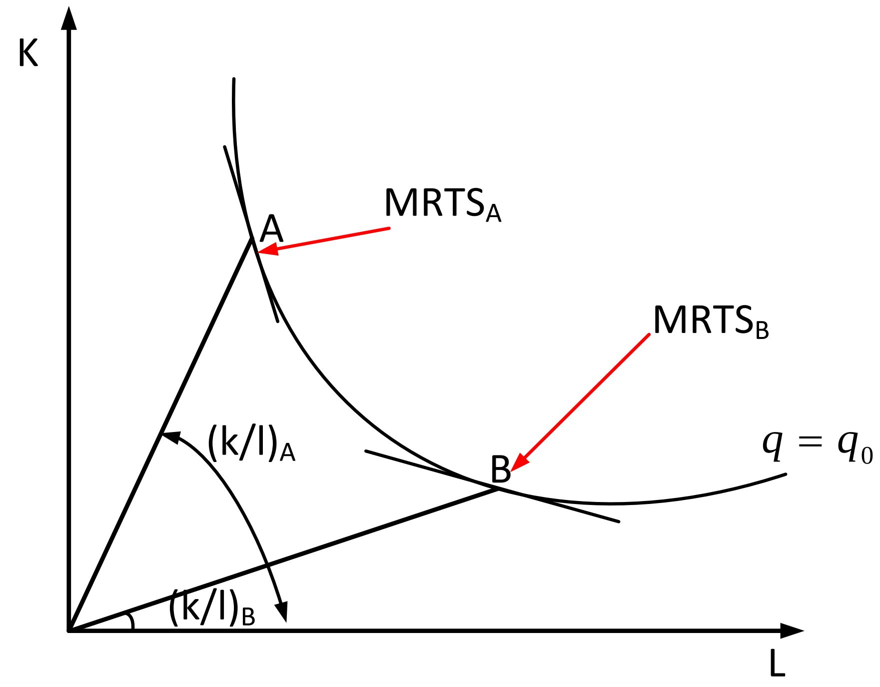
Elasticity of Substitution

- High elasticity of substitution (σ):
  - $MRTS_{l,k}$ does not change substantially relative to k/l.
  - Isoquant is relatively flat.



Elasticity of Substitution

- Low elasticity of substitution (σ):
  - $MRTS_{l,k}$ changes substantially relative to k/l.
  - Isoquant is relatively sharply curved.



Elasticity of Substitution: Linear Production Function

- Suppose that the production function is

$$q = f\,(k, l) = ak + bl$$

- This production function exhibits constant returns to scale

$$f\,(tk, tl) = atk + btl = t(ak + bl)$$
$$= tf(k, l)$$

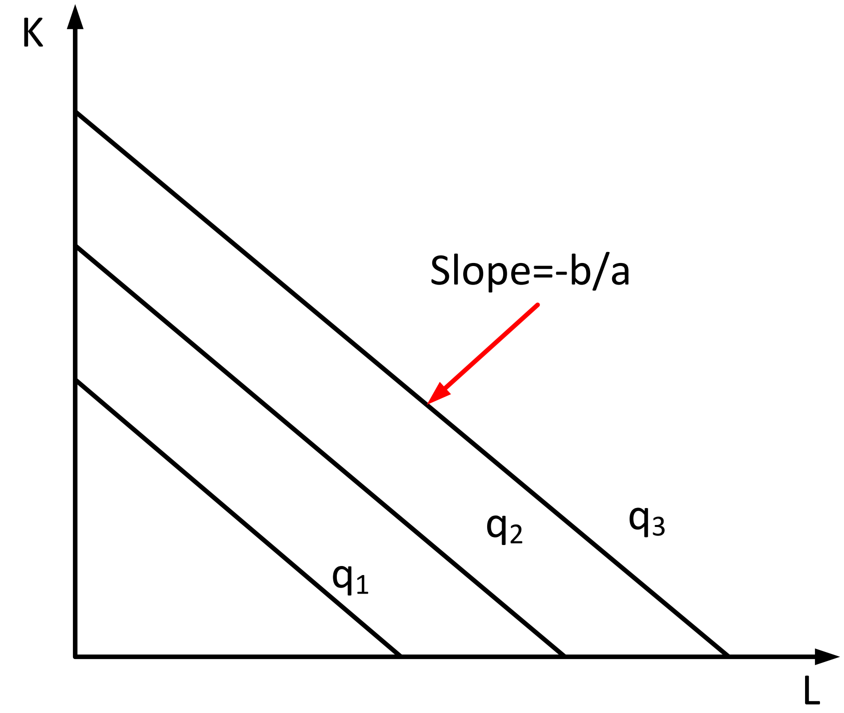
- Solving for k in q, we get $k = \frac{f\,(k,l)}{a} - \frac{b}{a}l$.
  - All isoquants are straight lines
  - k and l are perfect substitutes

Elasticity of Substitution: Linear Production Function

- $MRTS$ (slope of the isoquant) is constant as k/l changes.

$$\sigma = \frac{\%\Delta(k/l)}{\underbrace{\%\Delta MRTS_{l,k}}_{0}} = \infty$$

- This production function satisfies homotheticity.

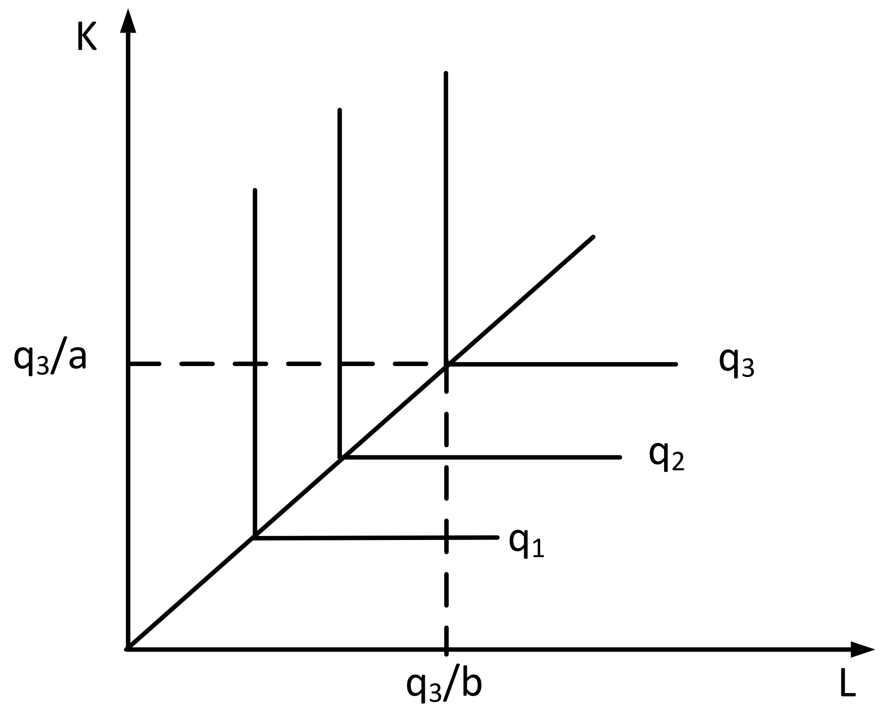


Elasticity of Substitution: Fixed Proportions Production Function

- Suppose that the production function is
$$q = \min(ak, bl) \quad a, b > 0$$
- Capital and labor must always be used in a fixed ratio
  - No substitution between k and l
  - The firm will always operate along a ray where k/l is constant (i.e., at the kink!).
- Because k/l is constant (b/a),
$$\sigma = \frac{\%\Delta(k/l)}{\underbrace{\%\Delta MRTS_{l,k}}_{\infty}} = 0$$

Elasticity of Substitution: Fixed Proportions Production Function

- $MRTS_{l,k} = \infty$ for l below the kink of the isoquant.
- $MRTS_{l,k} = 0$ for l beyond the kink.
- This production function also satisfies homotheticity.



Elasticity of Substitution: Cobb-Douglas Production Function

- Suppose that the production function is

$$q = f(k,l) = Ak^{a}l^{b} \quad A,a,b > 0$$

- This production function can exhibit any returns to scale

$$f(tk,tl) = A(tk)^{a}(tl)^{b} = At^{a+b}k^{a}l^{b}$$
$$= t^{a+b}f(k,l)$$

  - If $a + b = 1 \Longrightarrow$ constant returns to scale
  - If $a + b > 1 \Longrightarrow$ increasing returns to scale
  - If $a + b < 1 \Longrightarrow$ decreasing returns to scale

Elasticity of Substitution: Cobb-Douglas Production Function

- The Cobb-Douglass production function is linear in logarithms

$$\ln(q) = \ln(A) + a\ln(k) + b\ln(l)$$

  - a is the elasticity of output with respect to k

$$\varepsilon_{q,k} = \frac{\partial \ln(q)}{\partial \ln(k)}$$

  - b is the elasticity of output with respect to l

$$\varepsilon_{q,l} = \frac{\partial \ln(q)}{\partial \ln(l)}$$

Elasticity of Substitution: Cobb-Douglas Production Function

- The elasticity of substitution (σ) for the Cobb-Douglas production function:
  - First,

$$MRTS_{l,k} = \frac{MP_l}{MP_k} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial q}{\partial k}} = \frac{bAk^a l^{b-1}}{aAk^{a-1} l^b} = \frac{b}{a} \cdot \frac{k}{l}$$

  - Hence,

$$\ln(MRTS_{l,k}) = \ln\left(\frac{b}{a}\right) + \ln\left(\frac{k}{l}\right)$$

Elasticity of Substitution: Cobb-Douglas Production Function

– Solving for $\ln\left(\frac{k}{l}\right)$,

$$\ln\left(\frac{k}{l}\right) = \ln(MRTS_{l,k}) - \ln\left(\frac{b}{a}\right)$$

– Therefore, the elasticity of substitution between k and l is

$$\sigma = \frac{d\ln\left(\frac{k}{l}\right)}{d\ln(MRTS_{l,k})} = 1$$

Elasticity of Substitution: CES Production Function

- Suppose that the production function is

$$q = f(k, l) = (k^\rho + l^\rho)^{\gamma/\rho}$$

where $\rho \leq 1, \rho \neq 0, \gamma > 0$

  - $\gamma = 1 \Longrightarrow$ constant returns to scale
  - $\gamma > 1 \Longrightarrow$ increasing returns to scale
  - $\gamma < 1 \Longrightarrow$ decreasing returns to scale

- Alternative representation of the CES function

$$f(k, l) = \left(ak^{\frac{\sigma-1}{\sigma}} + bl^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma-1}{\sigma}}$$

where σ is the elasticity of substitution.

Elasticity of Substitution: CES Production Function

- The elasticity of substitution (σ) for the CES production function:
  - First,

$$MRTS_{l,k} = \frac{MP_l}{MP_k} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial q}{\partial k}} = \frac{\frac{\gamma}{\rho}[k^{\rho} + l^{\rho}]^{\frac{\gamma}{\rho}-1}(\rho l^{\rho-1})}{\frac{\gamma}{\rho}[k^{\rho} + l^{\rho}]^{\frac{\gamma}{\rho}-1}(\rho k^{\rho-1})}$$

$$= \left(\frac{l}{k}\right)^{\rho-1} = \left(\frac{k}{l}\right)^{1-\rho}$$

Elasticity of Substitution: CES Production Function

– Hence,

$$\ln(MRTS_{l,k}) = (\rho - 1)\ln\left(\frac{k}{l}\right)$$

– Solving for $\ln\left(\frac{k}{l}\right)$,

$$\ln\left(\frac{k}{l}\right) = \frac{1}{\rho - 1}\ln(MRTS_{l,k})$$

– Therefore, the elasticity of substitution between k and l is

$$\sigma = \frac{d\ln\left(\frac{k}{l}\right)}{d\ln(MRTS_{l,k})} = \frac{1}{\rho - 1}$$

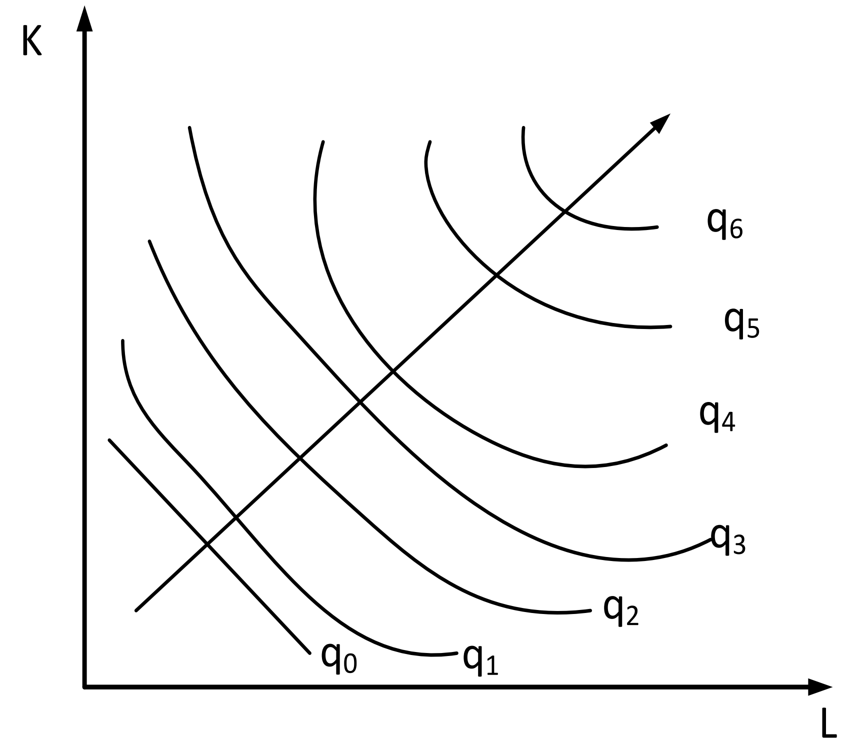
Elasticity of Substitution: CES Production Function

- Elasticity of Substitution in German Industries (Source: Kemfert, 1998):

Industry	σ
Food	0.66
Iron	0.50
Chemicals	0.37
Motor Vehicles	0.10

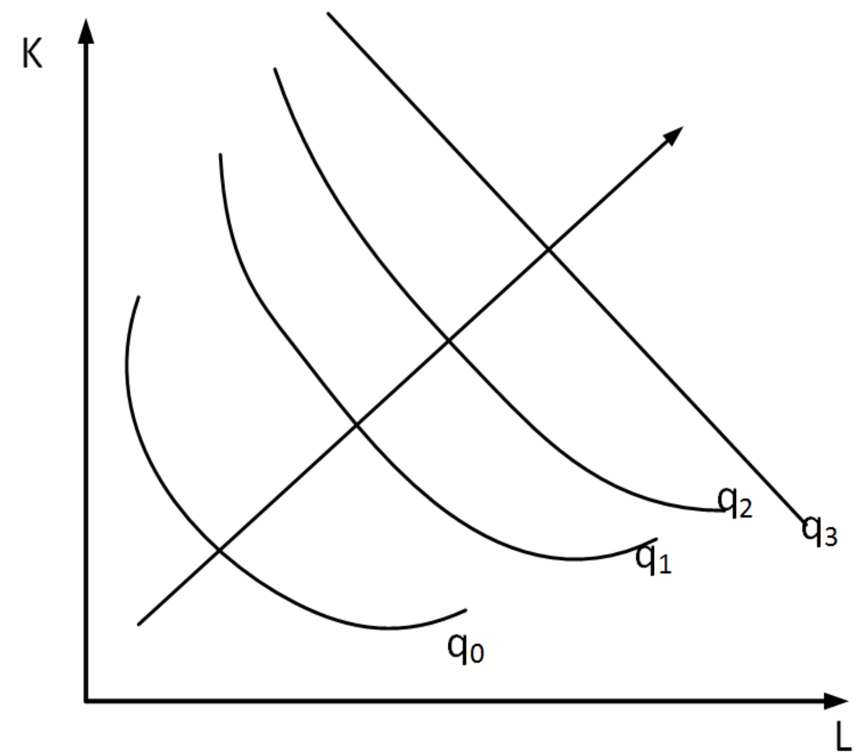
Elasticity of Substitution

- The elasticity of substitution σ between k and l is *decreasing* in scale (i.e., as q increases).
  - q_0 and q_1 have very high σ
  - q_5 and q_6 have very low σ



Elasticity of Substitution

- The elasticity of substitution σ between k and l is *increasing* in scale (i.e., as q increases).
  - q_0 and q_1 have very low σ
  - q_2 and q_3 have very high σ



Profit Maximization

Profit Maximization

- Assumptions:
  - Firms are price takers: the production plans of an individual firm do not alter price levels $p = (p_1, p_2, \dots, p_L) \gg 0$.
  - The production set satisfies: non-emptiness, closedness, and free-disposal.

- Profit maximization problem (PMP):

$$\max_{y} \quad p \cdot y$$

s.t. $y \in Y$, or alternatively, $F(y) \leq 0$

Profit Maximization

- Profit function $\pi(p)$ associates to every p the highest amount of profits (i.e., $\pi(p)$ is the value function of the PMP)

$$\pi(p) = \max_{y} \{p \cdot y : y \in Y\}$$

- And the supply correspondence $y(p)$ is the argmax of the PMP,

$$y(p) = \{y \in Y : p \cdot y = \pi(p)\}$$

  where ***positive*** components in the vector $y(p)$ is output supplied by the firm to the market, while ***negative*** components are inputs in its production process.

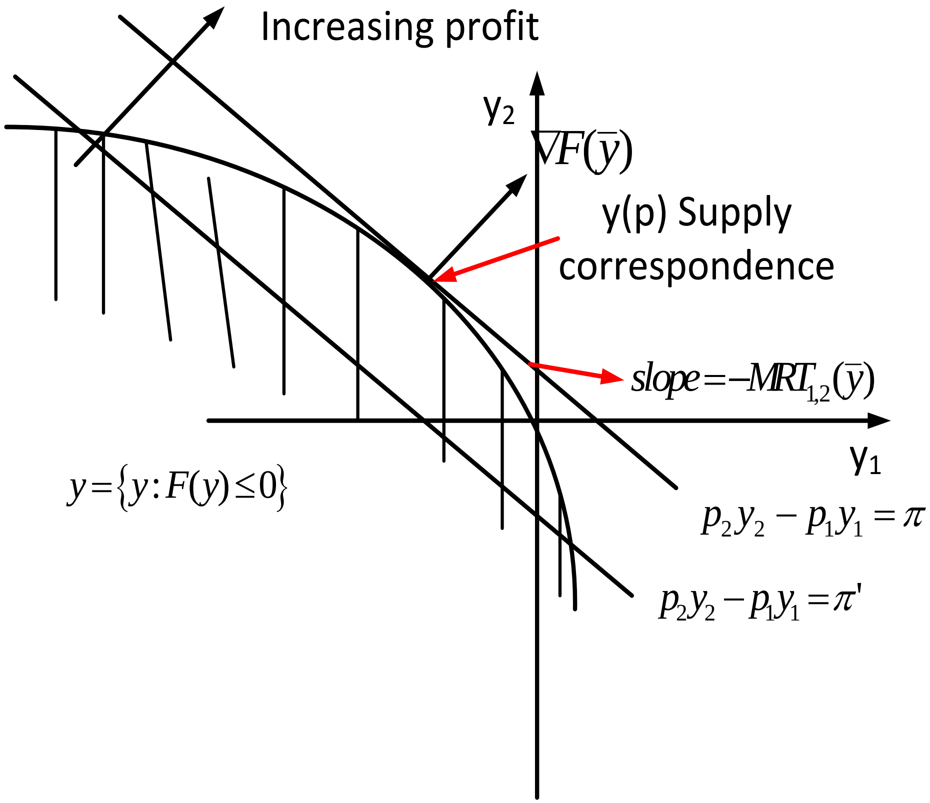
Profit Maximization

- ***Isoprofit line***: combinations of inputs and output for which the firm obtains a given level of profits.
- Note that

$$\pi^0 = p_2 y_2 - p_1 y_1$$

Solving for y_2

$$y_2 = \underbrace{\frac{\pi^0}{p_2}}_{\text{intercept}} \underbrace{-\frac{p_1}{p_2}}_{\text{slope}} y_1$$



Profit Maximization

- We can rewrite the PMP as

$$\max_{y \leq F(y)} \; p \cdot y$$

with associated Lagrangian

$$L = p \cdot y - \lambda F(y)$$

Profit Maximization

– Taking FOCs with respect to every y_k, we obtain

$$p_k - \lambda \frac{\partial F(y^*)}{\partial y_k} \leq 0$$

where $F(y^*)$ is evaluated at the optimum, i.e., $F(y^*) = F(y(p))$.

– For interior solutions, $p_k = \lambda \frac{\partial F(y^*)}{\partial y_k}$, or in matrix notation

$$p = \lambda \nabla_y F(y^*)$$

that is, the price vector and the gradient vector are proportional.

Profit Maximization

– Solving for λ, we obtain

$$\lambda = \frac{p_k}{\frac{\partial F(y^*)}{\partial y_k}} \text{ for every good } k \Longrightarrow \frac{p_k}{\frac{\partial F(y^*)}{\partial y_k}} = \frac{p_l}{\frac{\partial F(y^*)}{\partial y_l}}$$

which can also be expressed as

$$\frac{p_k}{p_l} = \frac{\frac{\partial F(y^*)}{\partial y_k}}{\frac{\partial F(y^*)}{\partial y_l}} \quad (= MRT_{k,l}(y^*))$$

– Graphically, the slope of the transformation frontier (at the profit maximization production plan y^*), $MRT_{k,l}(y^*)$, coincides with the price ratio, $\frac{p_k}{p_l}$.

Profit Maximization

- Are there PMPs with no supply correspondence $y(p)$, i.e., there is no well defined profit maximizing vector?
  - Yes.
- ***Example***: $q = f(z) = z$ (i.e., every unit of input z is transformed into a unit of output q)

Profit Maximization: Single Output

- Production function, $q = f(z)$, produces a single output from a vector z of inputs.

$$\max_{z \geq 0} \quad pf(z) - wz$$

- The first-order conditions are

$$p\frac{\partial f(y^*)}{\partial z_k} \leq w_k \text{ or } p \cdot MP_{z_k} \leq w_k$$

- For interior solutions, the market value of the marginal product obtained form using additional units of this input k, $p\frac{\partial f(y^*)}{\partial z_k}$, must coincide with the price of this input, w_k.

Profit Maximization: Single Output

- Note that for any two input, this implies

$$p = \frac{w_k}{\frac{\partial f(y^*)}{\partial z_k}} \text{ for every good } k$$

Hence,

$$\frac{w_k}{w_l} = \frac{\frac{\partial f(z^*)}{\partial z_k}}{\frac{\partial f(z^*)}{\partial z_l}} = \frac{MP_{z_k}}{MP_{z_l}} \;\; (= MRTS_{z_k, z_l}(z^*))$$

or

$$\frac{MP_{z_k}}{w_{z_k}} = \frac{MP_{z_l}}{w_{z_l}}$$

Intuition: Marginal productivity per dollar spent on input z_k is equal to that spent on input z_l.

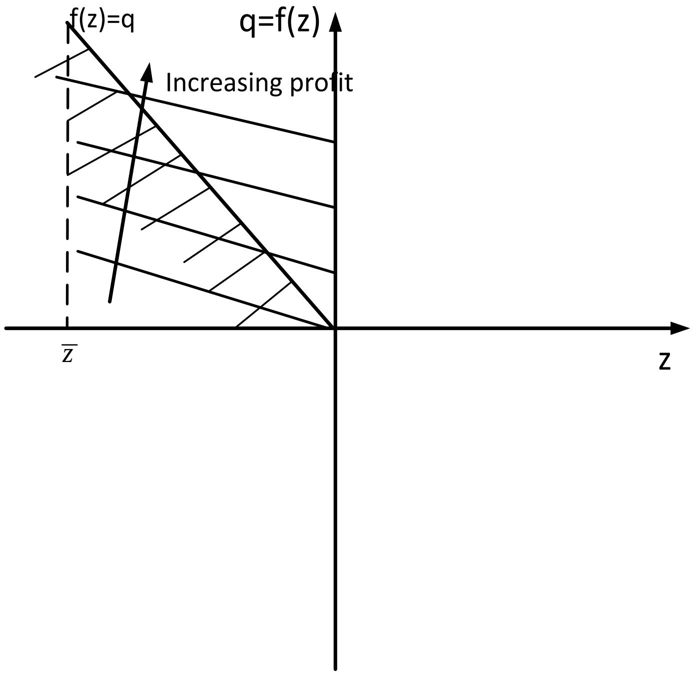
Profit Maximization

- ***Example*** : Are there PMPs with no supply correspondence $y(p)$, i.e., there is no well defined profit maximizing vector?
  - Yes.

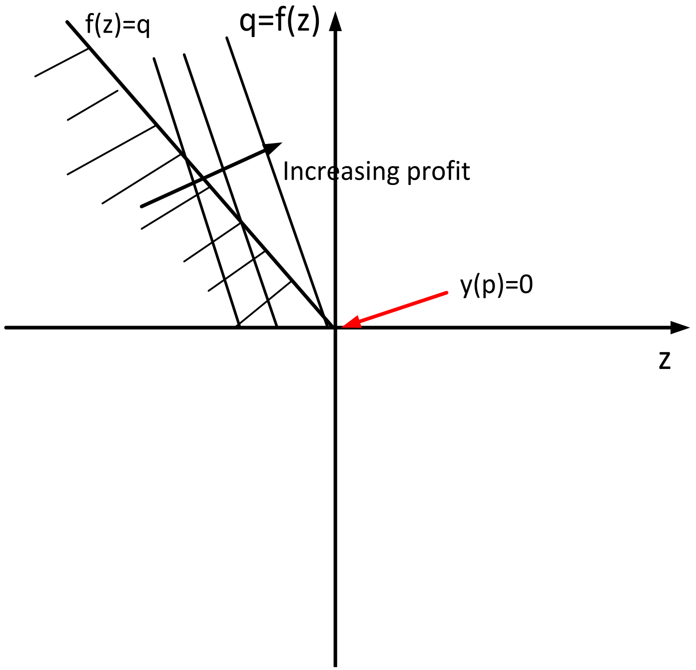
- If the input price p_z satisfies $p_z \geq p$, then $q = 0$ and $\pi(p) = 0$.
- If the input price p_z satisfies $p_z < p$, then $q = +\infty$ and $\pi(p) = +\infty$.
  - In this case, the supply correspondence is not well defined, since you can always increase input usage, thus increasing profits.
  - *Exception*: if input usage is constrained in the interval $[0, \bar{z}]$, then $y(p)$ is at the corner solution $y(p) = \bar{z}$, thus implying that the PMP is well defined.

Profit Maximization

- ***Example*** (continued):



If $p_z < p$, the firm can Δq and $\Delta\pi$.



If $p_z > p$, the firm chooses $q = y(p) = 0$ with $\pi(p) = 0$.

Profit Maximization: Single Output

- When are these FOCs also sufficient?
  - When the production set Y is *convex*! Let's see.
- Isocost line for the firm is

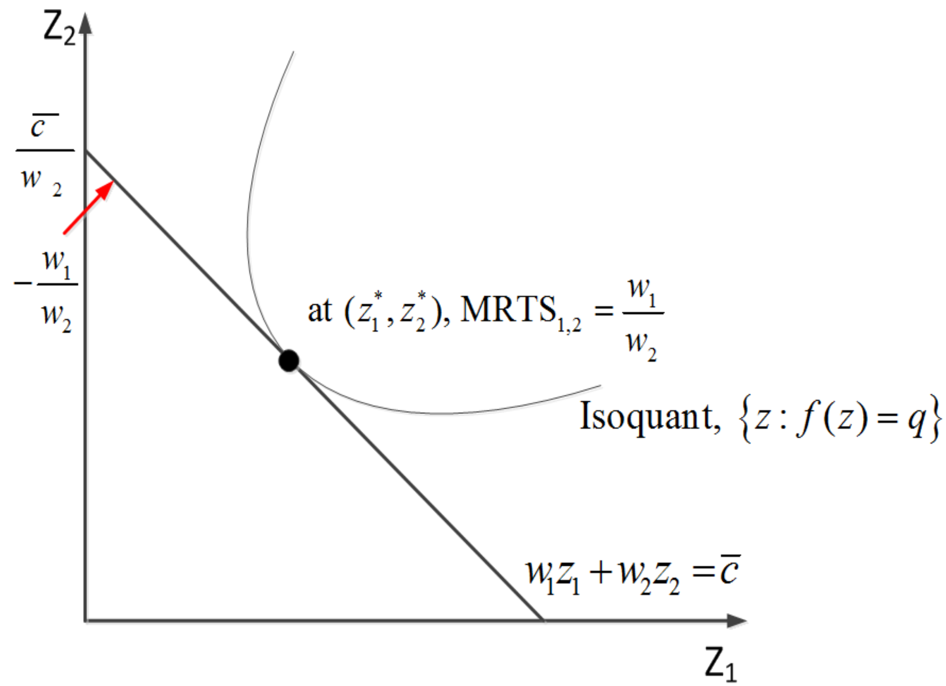
$$w_1 z_1 + w_2 z_2 = \bar{c}$$

- Solving for z_2

$$z_2 = \underbrace{\frac{\bar{c}}{w_2}}_{\text{intercept}} \underbrace{-\frac{w_1}{w_2}}_{\text{slope}} z_1$$

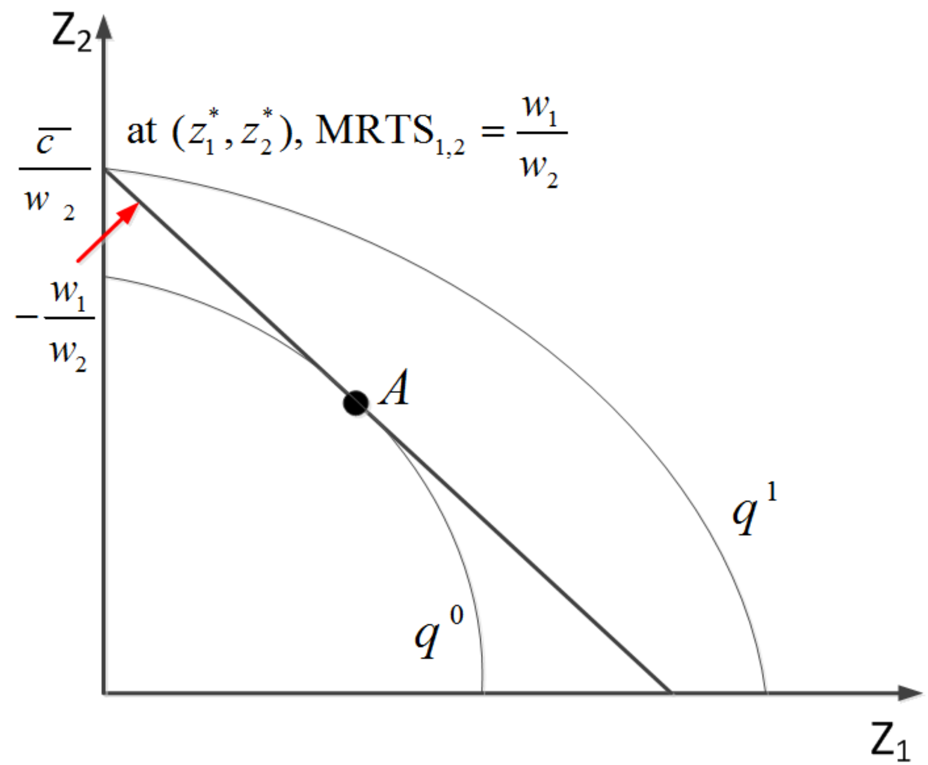
Profit Maximization: Single Output

- Convex production set
  - the FOCs (necessary) of $MRTS_{1,2} = \frac{w_1}{w_2}$ are also sufficient.



Profit Maximization: Single Output

- Non-convex production set
  - the FOCs are NOT sufficient for a combination of (z_1, z_2) that maximize profits.
  - the profit-maximizing vector (z_1^*, z_2^*) is at a corner solution, where the firm uses z_2 alone.



Profit Maximization: Single Output

- ***Example:*** Cobb-Douglas production function
- On your own:
  - Solve PMP (differentiating with respect to z_1 and z_2.
  - Find optimal input usage $z_1(w,q)$ and $z_2(w,q)$.
    - These are referred to as "conditional factor demand correspondences"
  - Plug them into the production function to obtain the value function, i.e., the output that arises when the firm uses its profit-maximizing input combination.

Properties of Profit Function

- Assume that the production set Y is closed and satisfies the free disposal property.

1) Homog(1) in prices

$$\pi(\lambda p) = \lambda\pi(p)$$

- Increasing the prices of all inputs and outputs by a common factor λ produces a proportional increase in the firm's profits.

$$\pi(p) = pq - w_1 z_1 - \cdots - w_n z_n$$

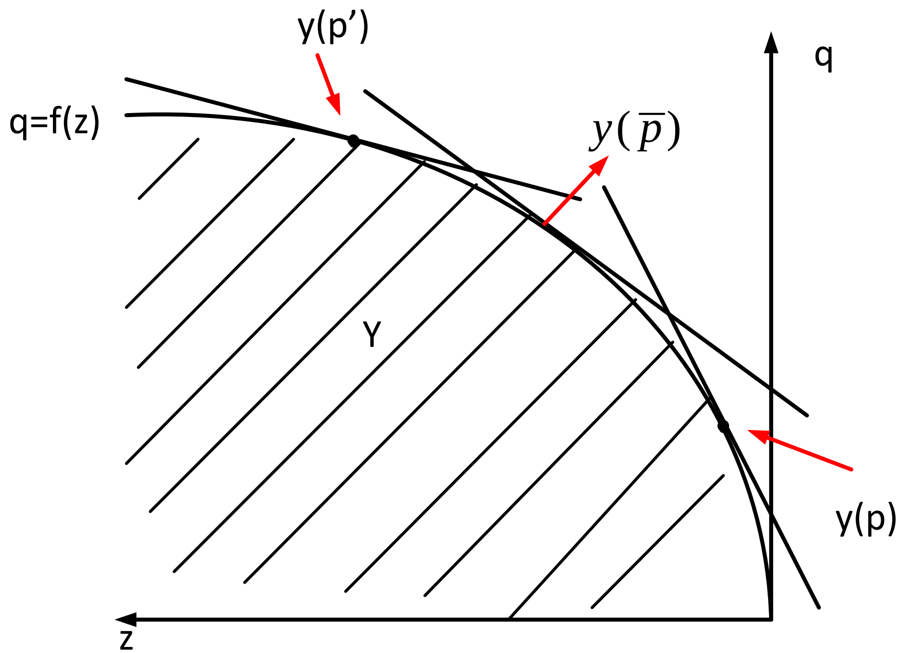
Scaling all prices by a common factor, we obtain

$$\pi(\lambda p) = \lambda pq - \lambda w_1 z_1 - \cdots - \lambda w_n z_n$$
$$= \lambda(pq - w_1 z_1 - \cdots - w_n z_n) = \lambda\pi(p)$$

Properties of Profit Function

2) Convex in output prices

- *Intuition*: the firm obtains more profits from balanced input-output combinations, than from unbalanced combinations.



Price vector	Production plan	Profits
p	$y(p)$	$\pi(p)$
p'	$y(p')$	$\pi(p')$
$\bar{p}$	$y(\bar{p})$	$\pi(\bar{p}) = \alpha\pi(p) + (1-\alpha)\pi(p')$

Properties of Profit Function

3) *If the production set Y is convex, then*
$$Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$$

- *Intuition*: the production set Y can be represented by this "dual" set.
- This dual set specifies that, for any given prices p, all production vectors y generate less profits $p \cdot y$, than the optimal production plan $y(p)$ in the profit function $\pi(p) = p \cdot y(p)$.

Properties of Profit Function

- All production plans (z, q) below the isoprofit line yield a lower profit level:

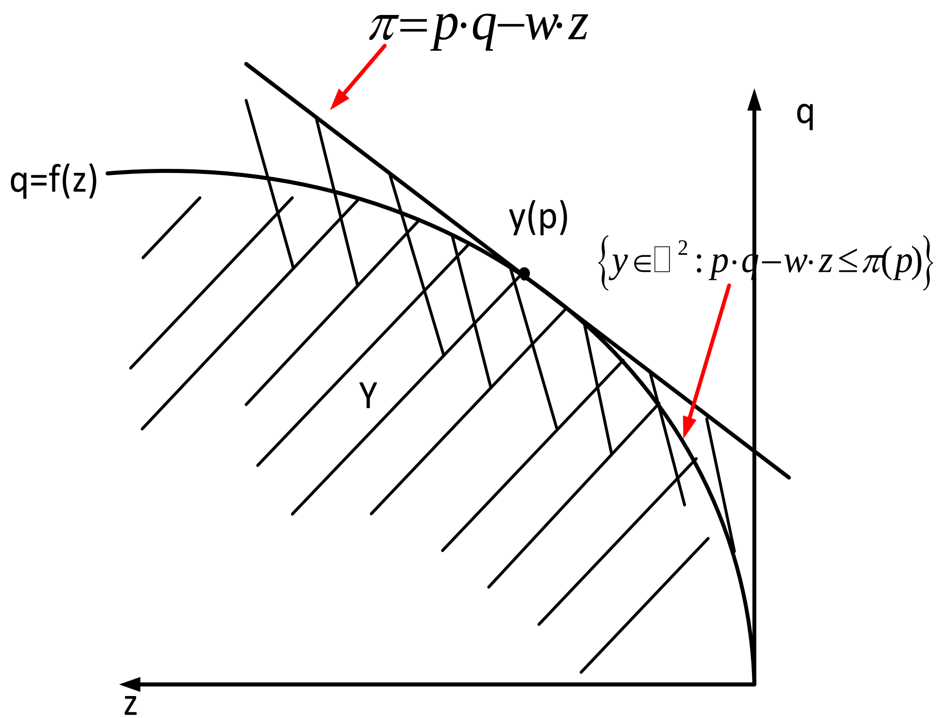
$$pq - wz \leq \pi(p)$$

- The isoprofit line $\pi(p) = pq - wz$ can be expressed as

$$q = \frac{\pi}{p} + \frac{w}{p} z$$

  – If $\frac{w}{p}$ is constant $\Longrightarrow \pi(\cdot)$ is convex.

  – What if it is not constant? Let's see next.



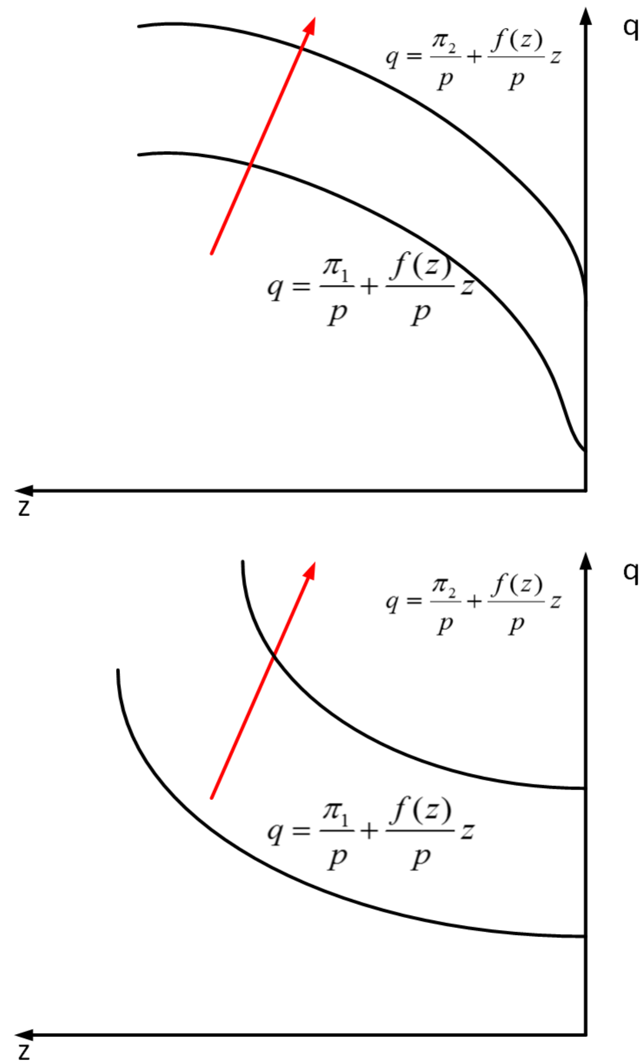
Properties of Profit Function

a) Input prices are a function of input usage, i.e., $w = f(z)$, where $f'(z) \neq 0$. Then, either
   i. $f'(z) < 0$, and the firm gets a *price discount* per unit of input from suppliers when ordering large amounts of inputs (e.g., loans)
   ii. $f'(z) > 0$, and the firm has to pay *more* per unit of input when ordering large amounts of inputs (e.g., scarce qualified labor)

b) Output prices are a function of production , i.e., $p = g(q)$, where $g'(q) \neq 0$. Then, either
   i. $g'(q) < 0$, and the firm offers *price discounts* to its customers.
   ii. $g'(q) > 0$, and the firm applies *price surcharges* to its customers.

Properties of Profit Function

- If $f'(z) < 0$, then we have strictly *convex* isoprofit curves.
- If $f'(z) > 0$, then we have strictly *concave* isoprofit curves.
- If $f'(z) = 0$, then we have *straight* isoprofit curves.



Remarks on Profit Function

- ***Remark 1***: the profit function is a value function, measuring firm profits *only* for the profit-maximizing vector y^*.
- ***Remark 2***: the profit function can be understood as a support function.
  – Take negative of the production set Y, i.e., $-Y$
  – Then, the support function of $-Y$ set is
  $$\mu_{-Y}(p) = \min_{y} \{p \cdot (-y) \colon\ y \in Y\}$$
  That is, take the profits resulting form all production vectors $y \in Y$, $p \cdot y$, then take the negative of all these profits, $p \cdot (-y)$, and then choose the smallest one.

Remarks on Profit Function

– Of course, this is the same as maximizing the (positive) value of the profits resulting from all production vector $y \in Y$, $p \cdot y$.

– Therefore, the profit function, $\pi(p)$, is the support of the negative production set, $-Y$,

$$\pi(p) = \mu_{-Y}(p)$$

Remarks on Profit Function

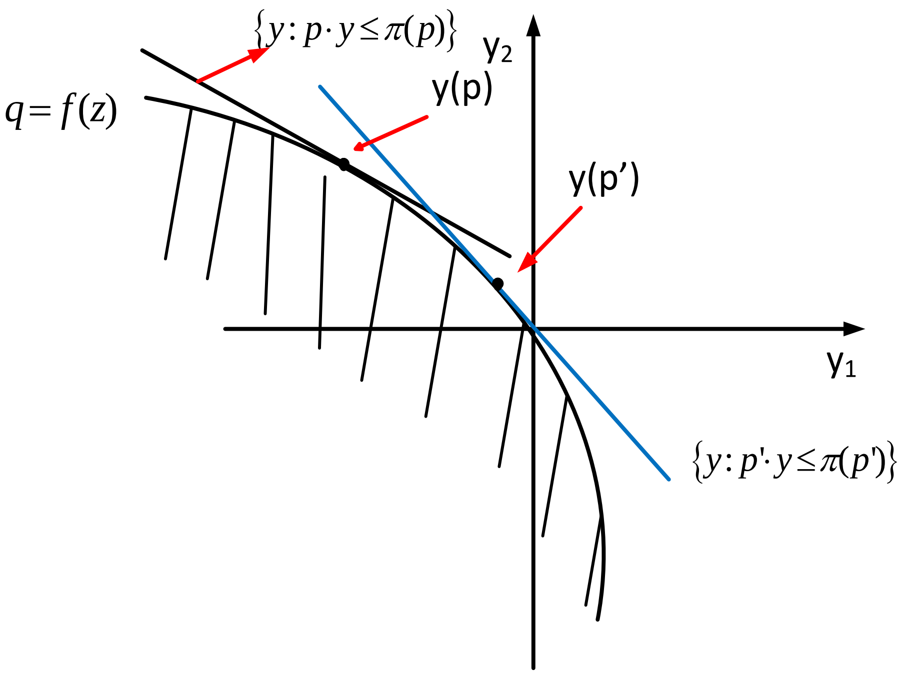
– Alternatively, the argmax of any objective function

$$y_1^* = \arg\max_y f(x)$$

coincides with the argmin of the negative of this objective function

$$y_2^* = \arg\max_y [-f(x)]$$

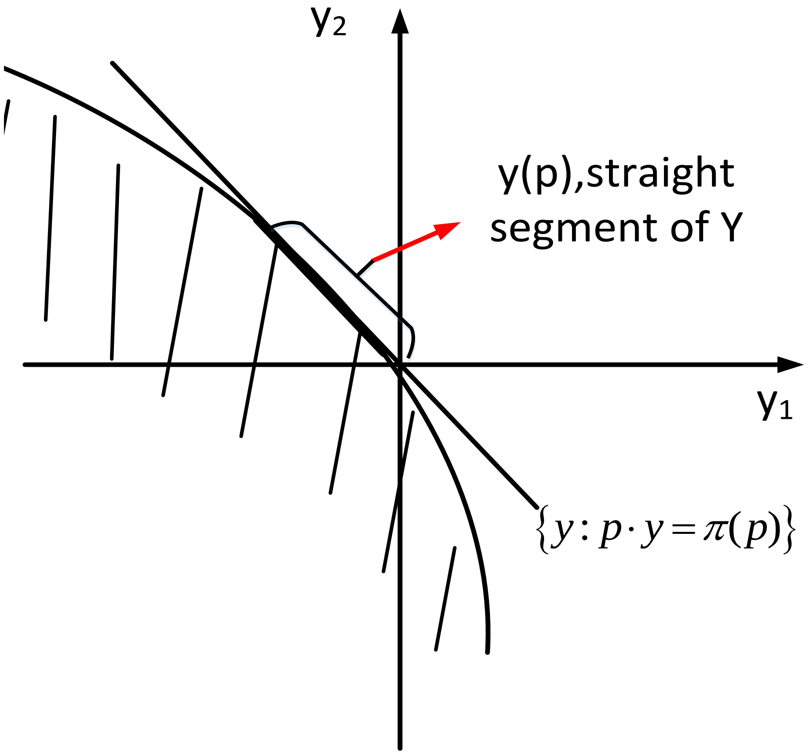
where $y_1^* = y_2^*$.



Properties of Supply Correspondence

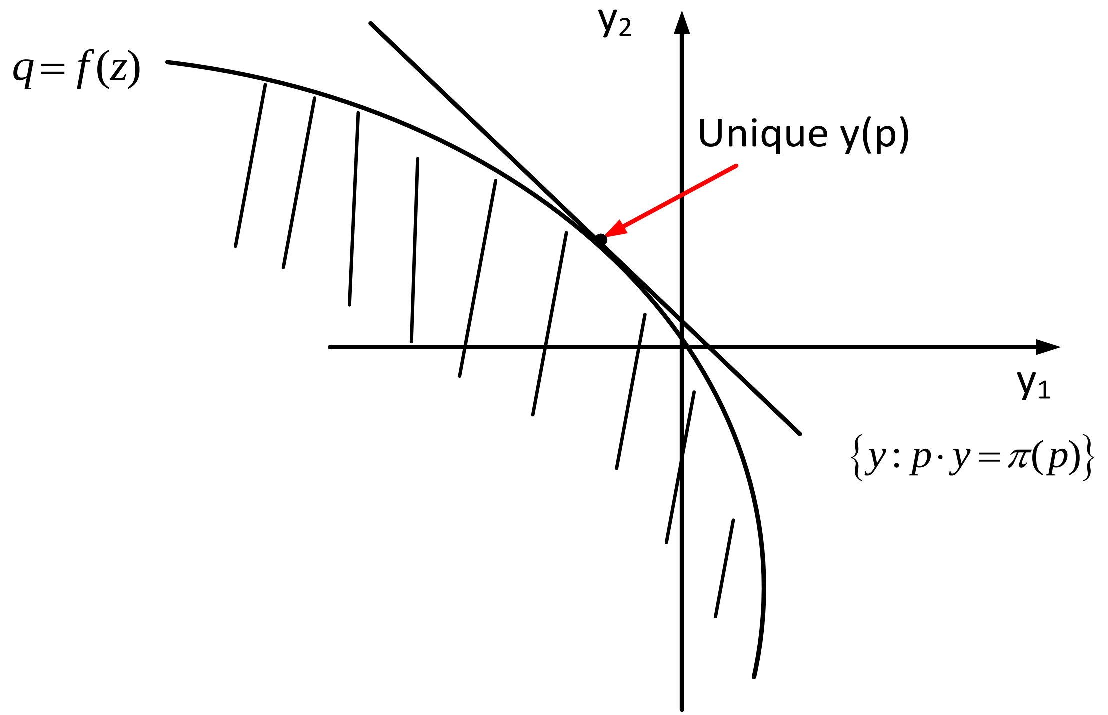
1) *If Y is weakly convex, then $y(p)$ is a convex set for all p.*

– Y has a flat surface
– $y(p)$ is NOT single valued.



Properties of Supply Correspondence

1) *(continued) If Y is strictly convex, then $y(p)$ is single-valued (if nonempty).*



Properties of Supply Correspondence

2) Hotelling's Lemma: If $y(\bar{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\bar{p}$. Moreover, $\nabla_p \pi(\bar{p}) = y(\bar{p})$.

- This is an application of the duality theorem from consumer theory.

- If $y(\cdot)$ is a function differentiable at $\bar{p}$, then $D_p y(\bar{p}) = D_p^2 \pi(\bar{p})$ is a symmetric and positive semidefinite matrix, with $D_p \pi(\bar{p})\bar{p} = 0$.
  - This is a direct consequence of the **law of supply**.

Properties of Supply Correspondence

– Since $D_p\pi(\bar{p})\bar{p} = 0, D_p\, y(\bar{p})$ must satisfy:

- *Own substitution effects* (main diagonal elements in $D_p y(\bar{p})$) are non-negative, i.e.,

$$\frac{\partial y_k(p)}{\partial p_k} \geq 0 \text{ for all } k$$

- *Cross substitution effects* (off diagonal elements in $D_p y(\bar{p})$) are symmetric, i.e.,

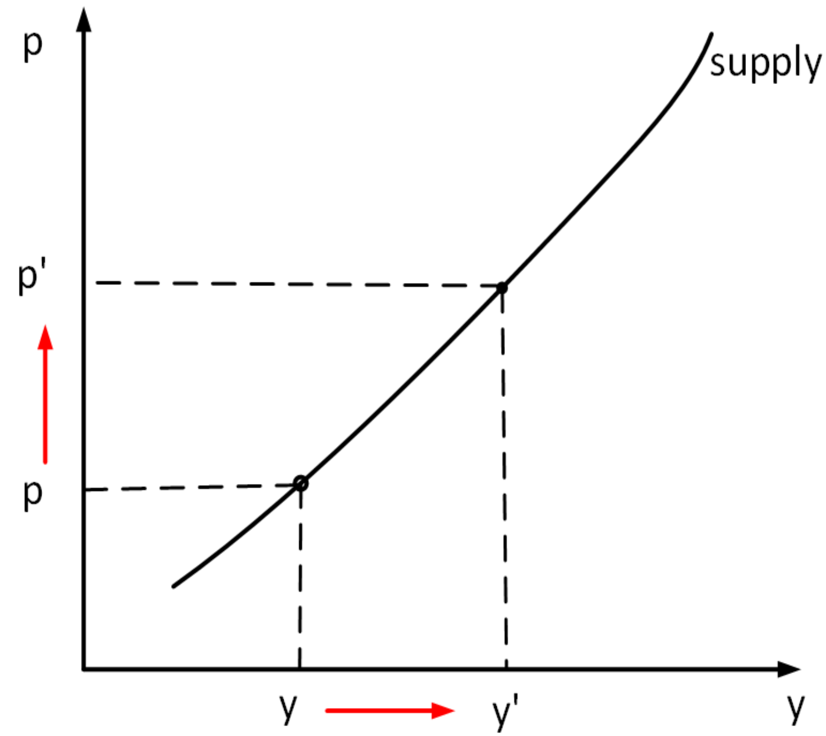
$$\frac{\partial y_l(p)}{\partial p_k} = \frac{\partial y_k(p)}{\partial p_l} \text{ for all } l \text{ and } k$$

Properties of Supply Correspondence

- $\frac{\partial y_k(p)}{\partial p_k} \geq 0$, which implies that quantities and prices move in the *same* direction,

  $(p - p')(y - y') \geq 0$

  – The **law of supply** holds!



Properties of Supply Correspondence

- Since there is no budget constraint, there is no wealth compensation requirement (as opposed to Demand theory).
  - This implies that there no income effects, only substitution effects.

- Alternatively, from a revealed preference argument, the law of supply can be expressed as

$$(p - p')(y - y') =$$
$$(py - py') + (p'y' - p'y) \geq 0$$

where $y \in y(p)$ and $y \in y(p')$.

Cost Minimization

Cost Minimization

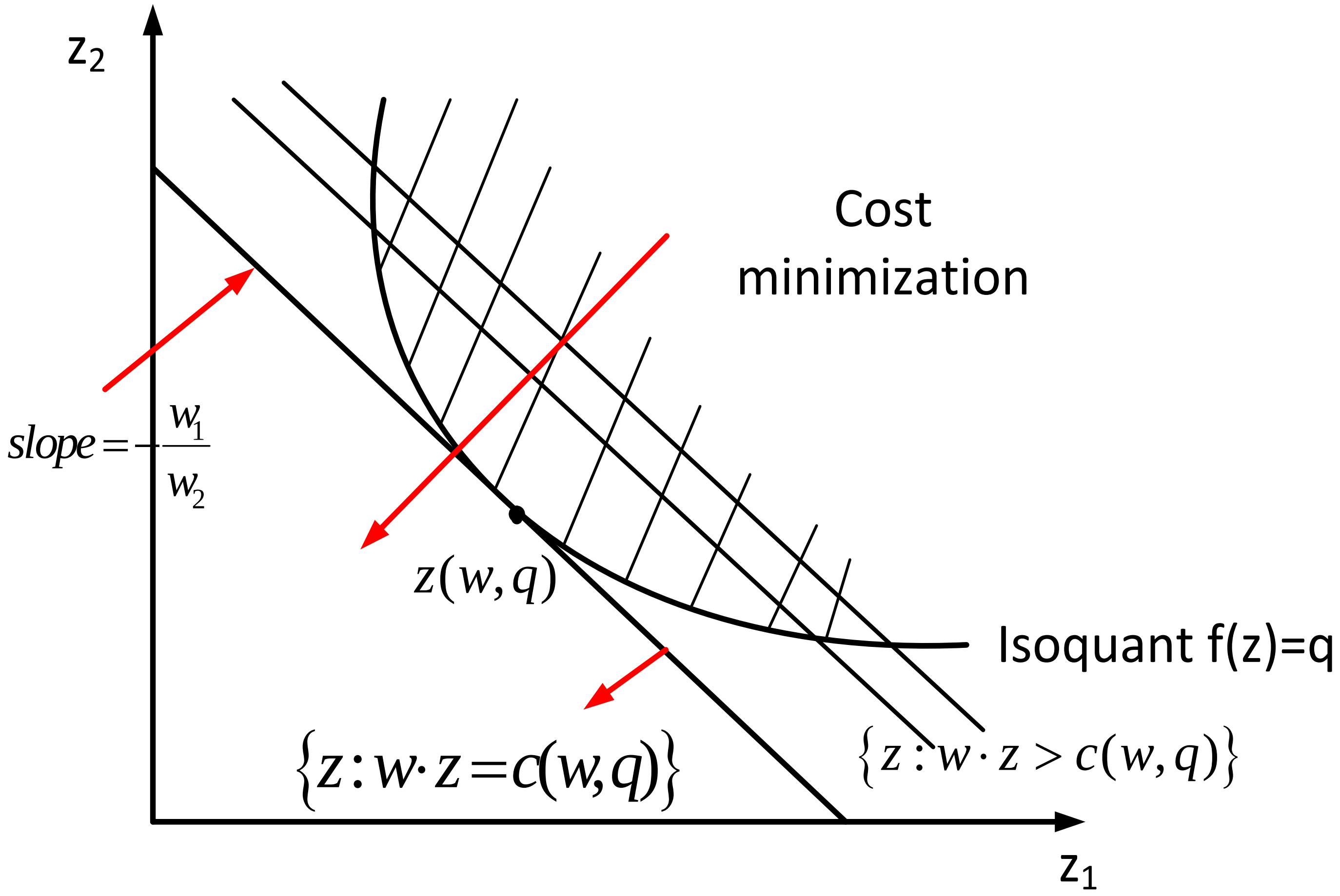
- We focus on the single output case, where
  - z is the input vector
  - $f(z)$ is the production function
  - q are the units of the (single) output
  - $w \gg 0$ is the vector of input prices
- The cost minimization problem (CMP) is

$$\min_{z \geq 0} \; w \cdot z$$
$$\text{s.t. } f(z) \geq q$$

Cost Minimization

- The optimal vector of input (or factor) choices is $z(w, q)$, and is known as the ***conditional factor demand correspondence***.
  - If single-valued, $z(w, q)$ is a function (not a correspondence)
  - Why "conditional"? Because it represents the firm's demand for inputs, conditional on reaching output level q.
- The value function of this CMP $c(w, q)$ is the ***cost function***.

Cost Minimization



Cost Minimization

- Graphically,
  - For a given isoquant $f(z) = q$, choose the isocost line associated with the lowest cost $w \cdot z$.
  - The tangency point is $z(w, q)$.
  - The isocost line associated with that combination of inputs is

$$\{z: w \cdot z = c(w, q)\}$$

    where the cost function $c(w, q)$ represents the lowest cost of producing output level q when input prices are w.
  - Other isocost lines are associated with either:
    - output levels higher than q (with costs exceeding $c(w, q)$), or
    - output levels lower than q (with costs below $c(w, q)$).

Cost Minimization

- The Lagrangian of the CMP is

$$\mathcal{L}(z;\lambda) = wz + \lambda[q - f(z)]$$

- Differentiating with respect to z_k

$$w_k - \lambda \frac{\partial f(z^*)}{\partial z_k} \geq 0$$

(= 0 if interior solution, z_k^*)

or in matrix notation

$$w - \lambda \nabla f(z^*) \geq 0$$

Cost Minimization

- From the above FOCs,

$$\frac{w_k}{\frac{\partial f(z^*)}{\partial z_k}} = \lambda \implies \frac{w_k}{w_l} = \frac{\frac{\partial f(z^*)}{\partial z_k}}{\frac{\partial f(z^*)}{\partial z_l}} \quad (= MRTS_{k,l}(z^*))$$

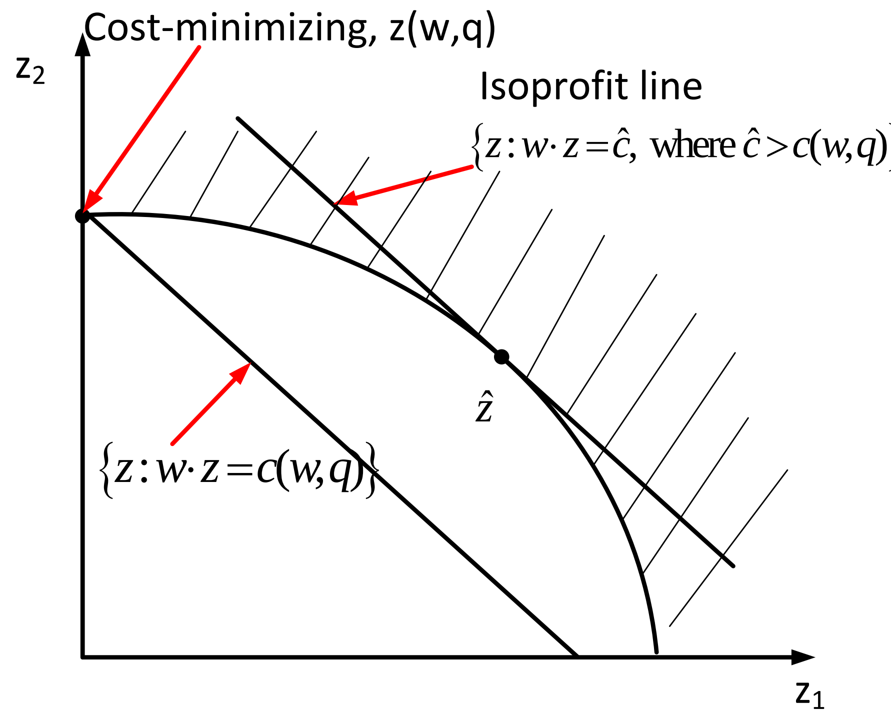
- Alternatively,

$$\frac{\frac{\partial f(z^*)}{\partial z_k}}{w_k} = \frac{\frac{\partial f(z^*)}{\partial z_l}}{w_l}$$

at the cost-minimizing input combination, the marginal product per dollar spent on input k must be equal that of input l.

Cost Minimization

- ***Sufficiency***: If the production set is *convex*, then the FOCs are also sufficient.
- A non-convex production set:
  - The input combinations satisfying the FOCs are NOT a cost-minimizing input combination $z(w, q)$.
  - The cost-minimizing combination of inputs $z(w, q)$ occurs at the corner.



Cost Minimization

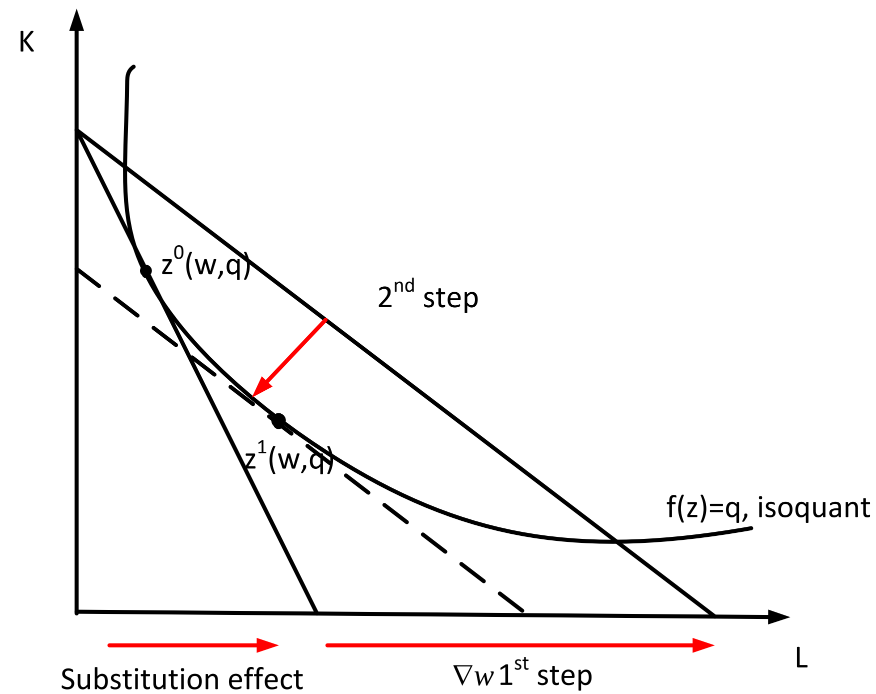
- ***Lagrange multiplier***: λ can be interpreted as the cost increase that the firm experiences when it needs to produce a higher level q.
  - Recall that, generally, the Lagrange multiplier represents the variation in the objective function that we obtain if we relax the constraint (e.g., wealth in UMP, utility level we must reach in the EMP).
- Therefore, λ is the marginal cost of production: the marginal increase in the firm's costs form producing additional units.

Cost Minimization: SE and OE Effects

- ***Comparative statics of*** $z(w, q)$: Let us analyze the effects of an input price change. Consider two inputs, e.g., labor and capital. When the price of labor, w, falls, two effects occur:
  - *Substitution effect*: if output is held constant, there will be a tendency for the firm to substitute l for k.
  - *Output effect*: a reduction in firm's costs allows it to produce larger amounts of output (i.e., higher isoquant), which entails the use of more units of l for k.

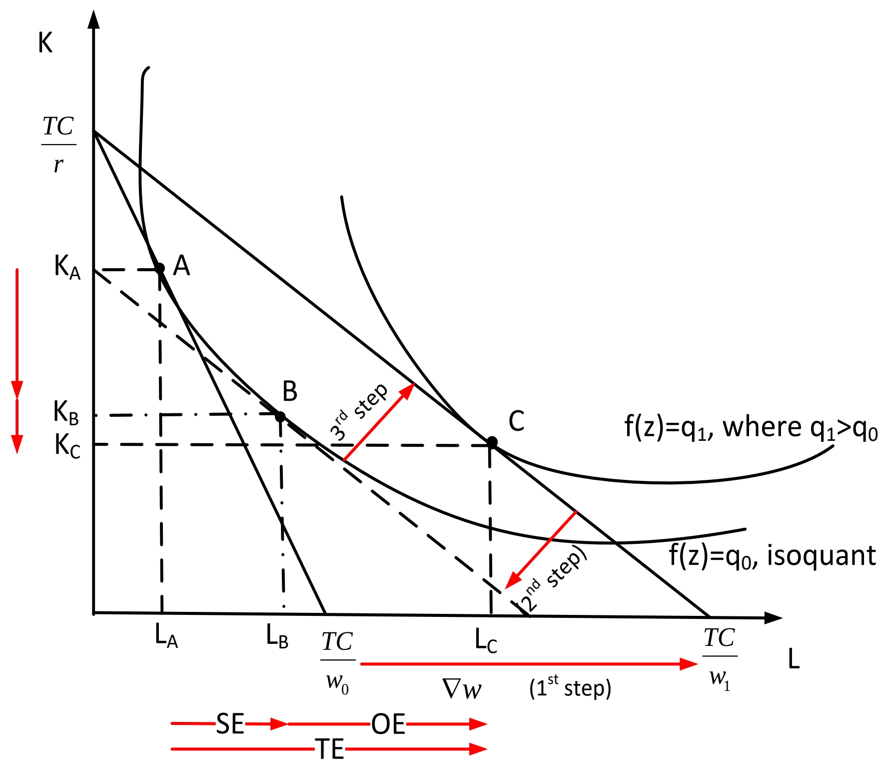
Cost Minimization: SE and OE Effects

- Substitution effect:
  - $z^0(w,q)$ solves CMP at the initial prices.
  - $\downarrow$ in wages $\implies$ isocost line pivots outwards.
  - To reach q, push the new isocost inwards in a parallel fashion.
  - $z^1(w,q)$ solves CMP at the new input prices (for output level q).
  - At $z^1(w,q)$, firm uses more l and less k.



Cost Minimization: SE and OE Effects

- Substitution effect (SE):
  - increase in labor demand from L_A to L_B.
  - same output as before the input price change.
- Output effect (OE):
  - increase in labor demand from L_B to L_C.
  - output level increases, total cost is the same as before the input price change.



Cost Minimization: Own-Price Effect

- We have two concepts of demand for any input
  - the conditional demand for labor, $l^c(r,w,q)$
    - $l^c(r,w,q)$ solves the CMP
  - the unconditional demand for labor, $l(p,r,w)$
    - $l(p,r,w)$ solves the PMP
- At the profit-maximizing level of output, i.e., $q(p,r,w)$, the two must coincide

$$l(p,r,w) = l^c(r,w,q) = l^c(r,w,q(p,r,w))$$

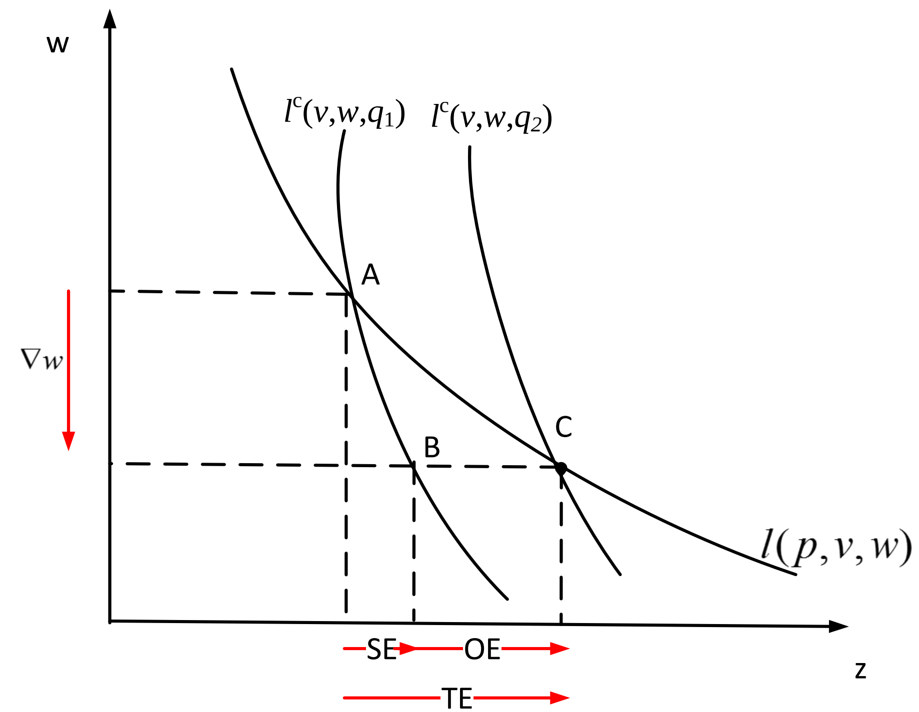
Cost Minimization: Own-Price Effect

- Differentiating with respect to w yields

$$\frac{\partial l(p,r,w)}{\partial w} = \underbrace{\underbrace{\frac{\partial l^c(r,w,q)}{\partial w}}_{SE\ (-)} + \underbrace{\overbrace{\frac{\partial l^c(r,w,q)}{\partial q}}^{(+)} \cdot \frac{\overbrace{\partial q}^{(-)}}{\partial w}}_{OE\ (-)}}_{TE\ (-)}$$

Cost Minimization: Own-Price Effect

- Since $TE > SE$, the unconditional labor demand is flatter than the conditional labor demand.
- Both SE and OE are negative.
  - Giffen paradox from consumer theory cannot arise in production theory.



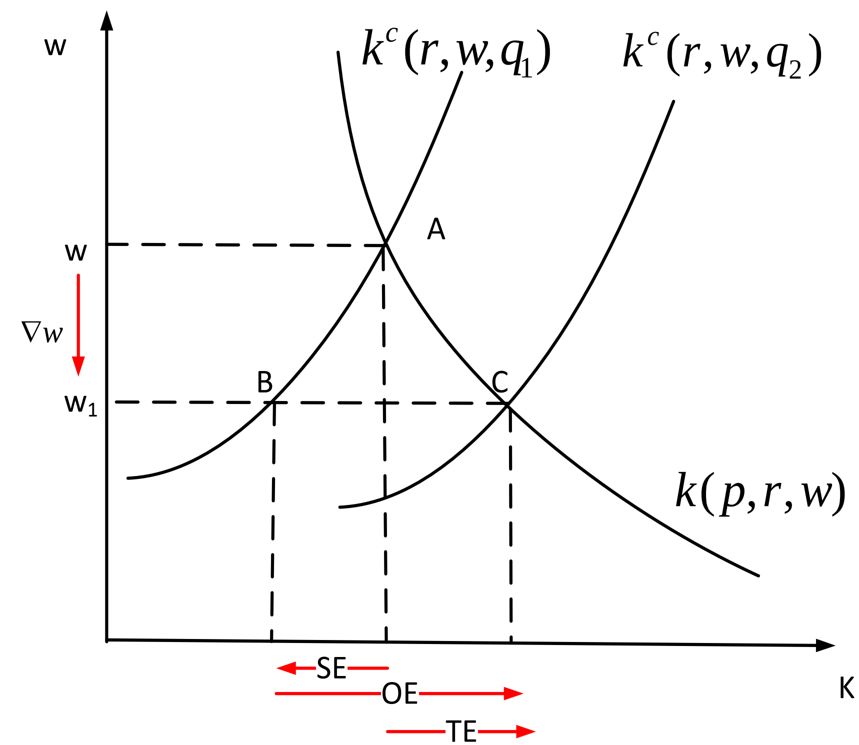
Cost Minimization: Cross-Price Effect

- No definite statement can be made about cross-price (CP) effects.
  - A fall in the wage will lead the firm to substitute away from capital.
  - The output effect will cause more capital to be demanded as the firm expands production.

$$\underbrace{\frac{\partial k(p,r,w)}{\partial w}}_{CP\ TE\ (+)\text{ or }(-)} = \underbrace{\frac{\partial k^c(r,w,q)}{\partial w}}_{CP\ SE\ (+)} + \underbrace{\frac{\overbrace{\partial k^c(r,w,q)}^{(+)}}{\partial q} \cdot \frac{\overbrace{\partial q}^{(-)}}{\partial w}}_{CP\ OE\ (-)}$$

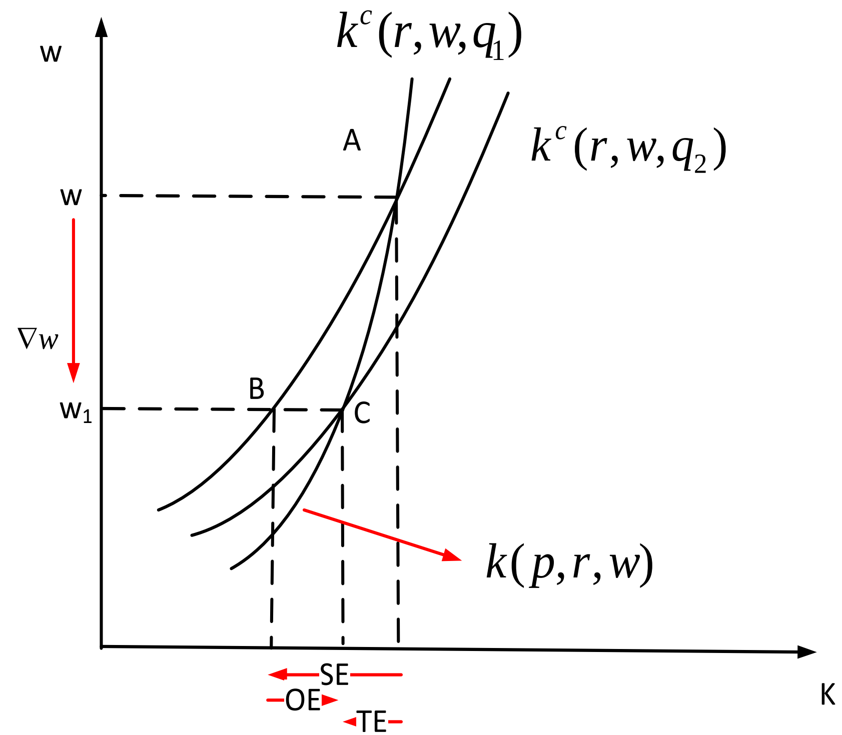
Cost Minimization: Cross-Price Effect

- The (+) cross-price OE completely offsets the (−) cross-price SE, leading to a positive cross-price TE.



Cost Minimization: Cross-Price Effect

- The $(+)$ cross-price OE only partially offsets the $(-)$ cross-price SE, leading to a negative cross-price TE.



Properties of Cost Function

- Assume that the production set Y is closed and satisfies the free disposal property.

1) *$c(w, q)$ is Homog(1) in w*

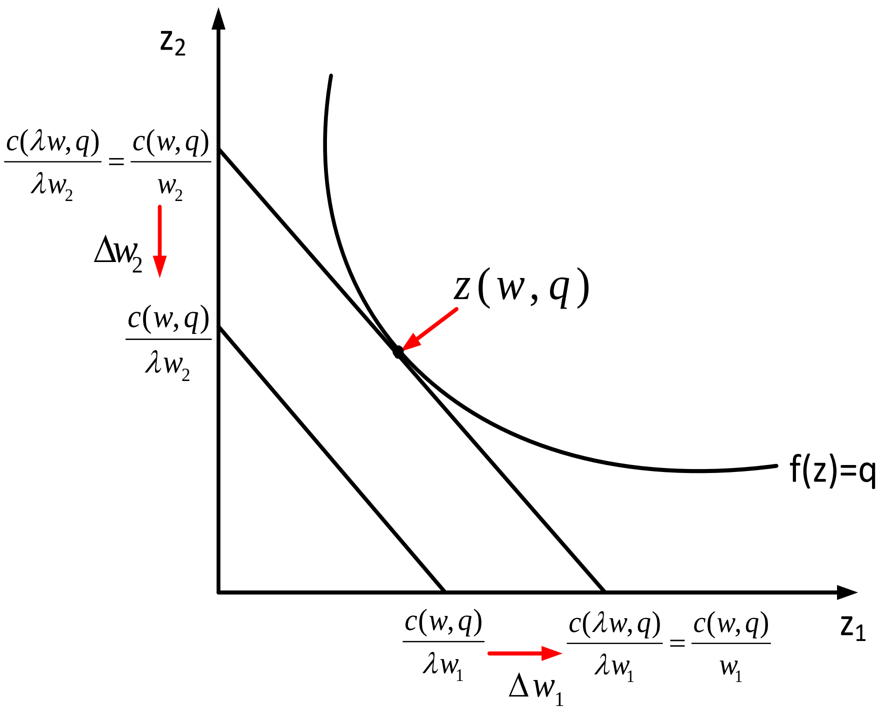
  - That is, increasing all input prices by a common factor λ yields a proportional increase in the minimal costs of production:

$$c(\lambda w, q) = \lambda c(w, q)$$

  since $c(w, q)$ represents the minimal cost of producing a given output q at input prices w.

Properties of Cost Function

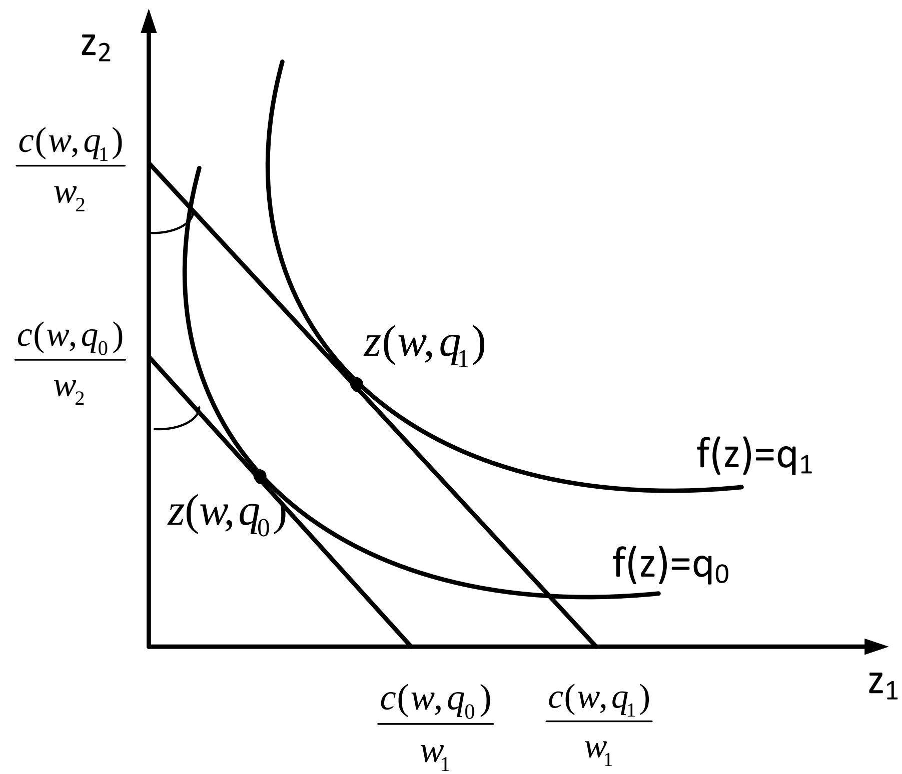
- An increase in all input prices (w_1, w_2) by the same proportion λ produces a parallel downward shift in the firm's isocost line.



Properties of Cost Function

2) $c(w, q)$ is non-decreasing in q.

- Producing higher output levels implies a weakly higher minimal cost of production
- If $q_1 > q_0$, then it must be
  $c(w, q_1) > c(w, q_0)$



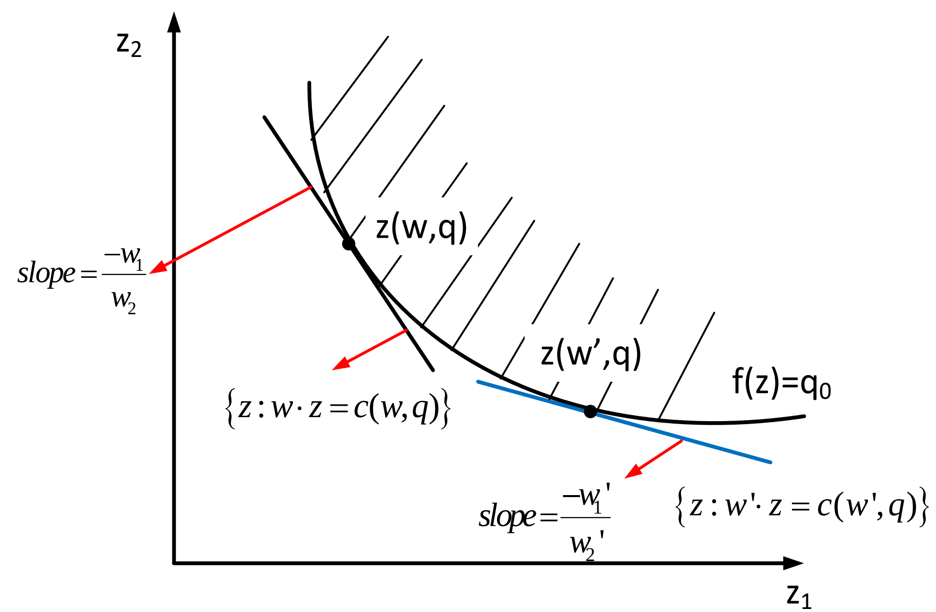
Properties of Cost Function

3) *If the set* $\{z \geq 0: f(z) \geq q\}$ *is convex for every* q*, then the production set can be described as*

$$Y = \left\{ \begin{matrix} (-z, q): w \cdot z \geq c(w, q) \\ \text{for every } \ w \gg 0 \end{matrix} \right\}$$

Properties of Cost Function

- Take $f(z) = q$.
- For input prices $w = (w_1, w_2)$, find $c(w, q)$ by solving CMP.
- For input prices $w' = (w_1', w_2')$, find $c(w', q)$ by solving CMP.
- The intersection of "more costly" input combinations $w \cdot z \geq c(w, q)$, for every input prices $w \gg 0$, describes set $f(z) \geq q$.

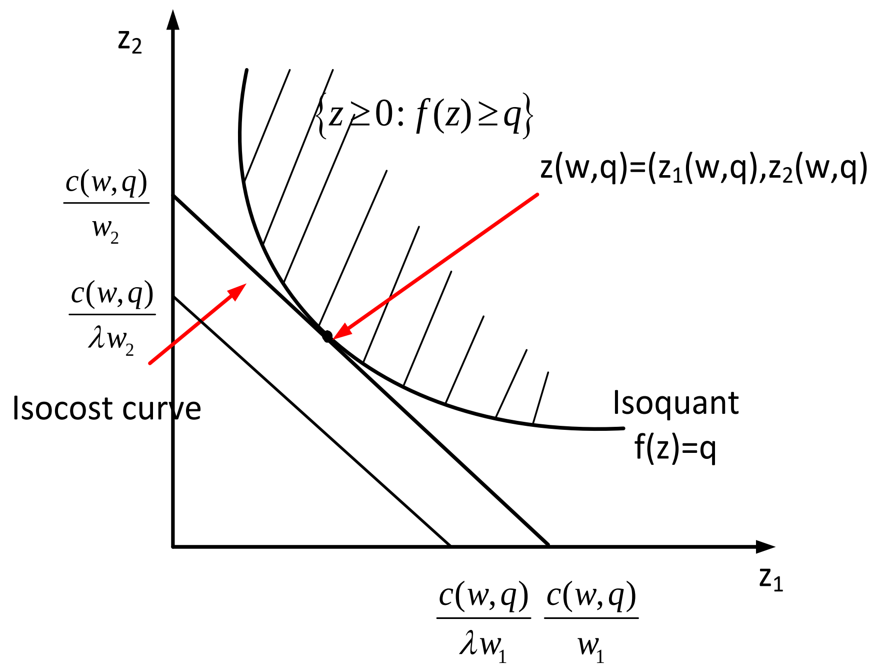


Properties of Conditional Factor Demand Correspondence

1) $z(w, q)$ *is Homog(0) in* w.

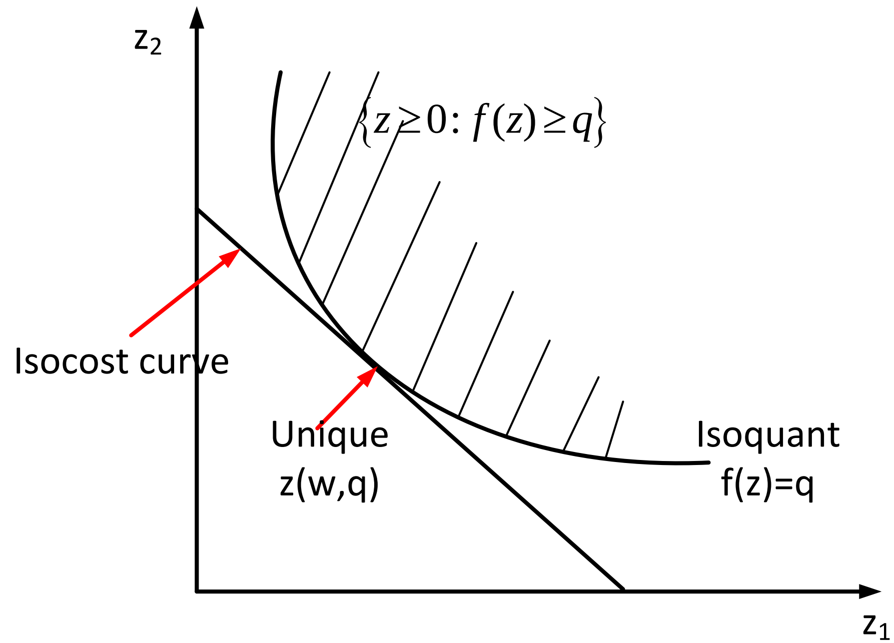
- That is, increasing input prices by the same factor λ does not alter the firm's demand for inputs at all,

$$z(\lambda w, q) = z(w, q)$$



Properties of Conditional Factor Demand Correspondence

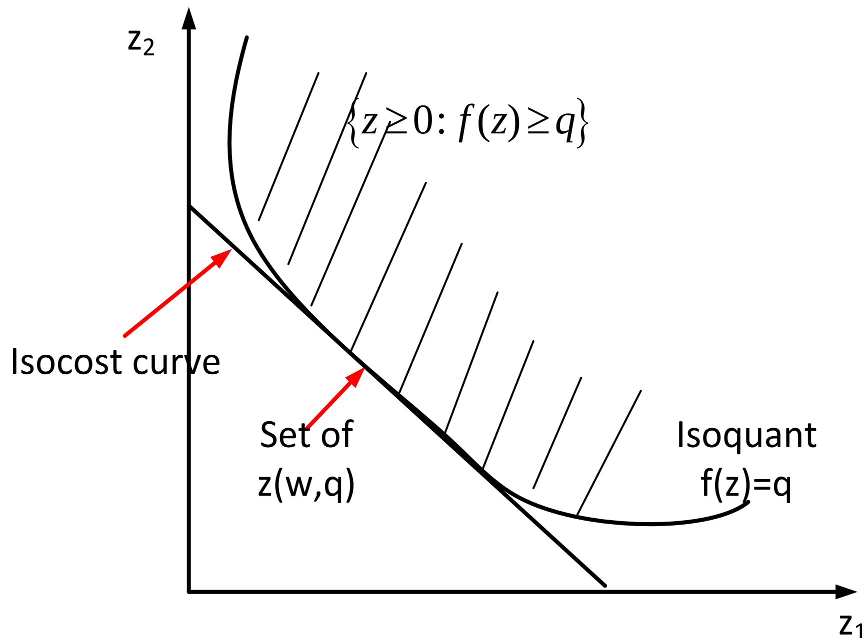
2) *If the set* $\{z \geq 0 : f(z) \geq q\}$ *is strictly convex, then the firm's demand correspondence* $z(w, q)$ *is single valued.*



Properties of Conditional Factor Demand Correspondence

2) *(continued)*

If the set $\{z \geq 0: f(z) \geq q\}$ *is weakly convex, then the demand correspondence* $z(w, q)$ *is not a single-valued, but a convex set.*



Properties of Conditional Factor Demand Correspondence

3) ***Shepard's lemma***: If $z(\bar{w},q)$ consists of a single point, then $c(w,q)$ is differentiable with respect to w at, $\bar{w}$, and

$$\nabla_w c(\bar{w},q) = z(\bar{w},q)$$

Properties of Conditional Factor Demand Correspondence

4) *If $z(w, q)$ is differentiable at $\overline{w}$, then $D_w^2 c(\overline{w}, q) = D_w z(\overline{w}, q)$ is a symmetric and negative semidefinite matrix, with $D_w z(\overline{w}, q) \cdot \overline{w} = 0$.*

- $D_w z(\overline{w}, q)$ is a matrix representing how the firm's demand for every unit responds to changes in the price of such input, or in the price of the other inputs.

Properties of Conditional Factor Demand Correspondence

4) (continued)

- Own substitution effects are non-positive,

$$\frac{\partial z_k(w,q)}{\partial w_k} \leq 0 \text{ for every input } k$$

i.e., if the price of input k increases, the firm's factor demand for this input decreases.

- Cross substitution effects are symmetric,

$$\frac{\partial z_k(w,q)}{\partial w_l} = \frac{\partial z_l(w,q)}{\partial w_k} \text{ for all inputs } k \text{ and } l$$

Properties of Production Function

1) *If $f(z)$ is Homog(1) (i.e., if $f(z)$ exhibits constant returns to scale), then $c(w,q)$ and $z(w,q)$ are Homog(1) in q.*

- Intuitively, if $f(z)$ exhibits CRS, then an increase in the output level we seek to reach induces an increase of the same proportion in the cost function and in the demand for inputs. That is,

$$c(w,\lambda q) = \lambda c(w,q)$$

and

$$z(w,\lambda q) = \lambda z(w,q)$$

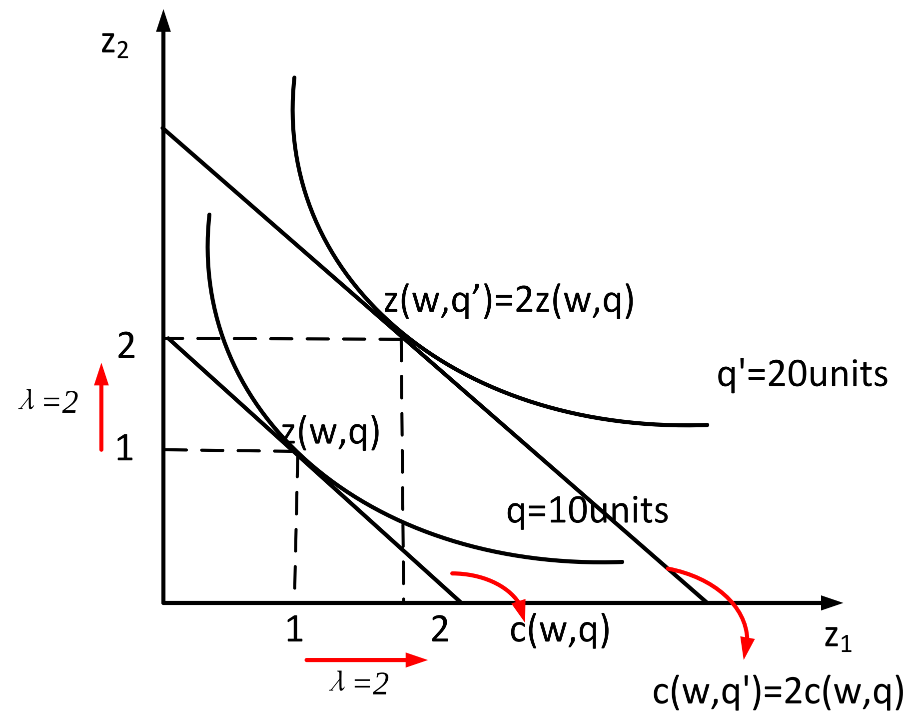
Properties of Production Function

- $\lambda = 2$ implies that demand for inputs doubles

  $$z(w, 2q) = 2z(w, q)$$

and that minimal costs also double

$$c(w, 2q) = 2c(w, q)$$



Properties of Production Function

2) *If $f(z)$ is concave, then $c(w, q)$ is convex function of q (i.e., marginal costs are non-decreasing in q).*

- More compactly,

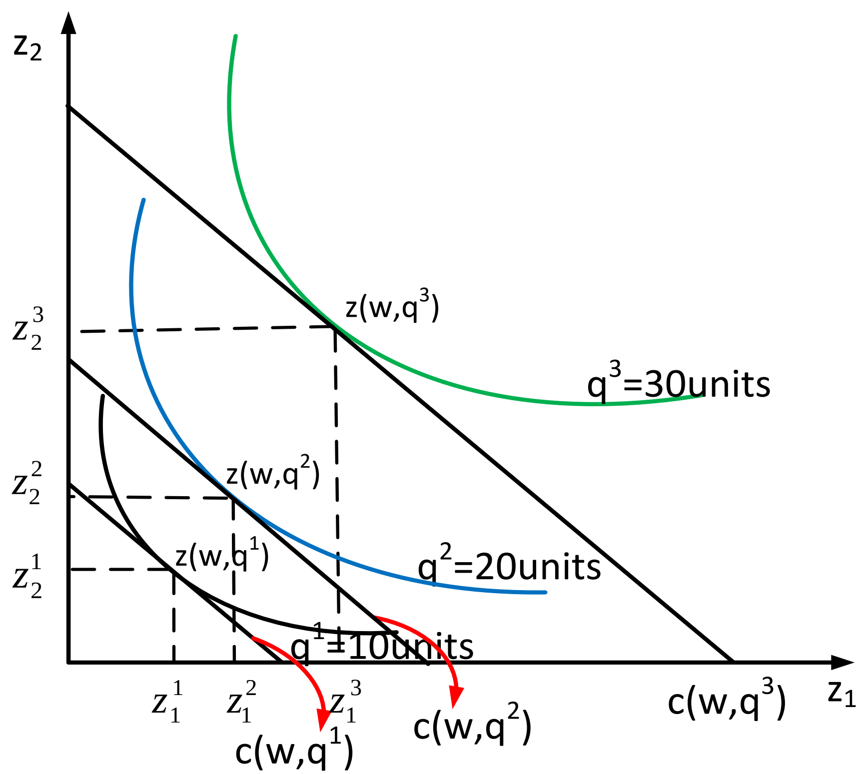
$$\frac{\partial^2 c(w,q)}{\partial q^2} \geq 0$$

or, in other words, marginal costs $\frac{\partial c(w,q)}{\partial q}$ are weakly increasing in q.

Properties of Production Function

2) (continued)

- Firm uses more inputs when raising output from q^2 to q^3 than from q^1 to q^2.
- Hence,
  $$c(w,q^3) - c(w,q^2) > c(w,q^2) - c(w,q^1)$$
- This reflects the convexity of the cost function $c(w,q)$ with respect to q.



Alternative Representation of PMP

Alternative Representation of PMP

- Using the cost function $c(w, q)$, we write the PMP as follows

$$\max_{q \geq 0} \quad pq - c(w, q)$$

This is useful if we have information about the cost function, but we don't about the production function $q = f(z)$.

Alternative Representation of PMP

- Let us now solve this alternative PMP

$$\max_{q \geq 0} \quad pq - c(w, q)$$

- FOCs for q^* to be profit maximizing are

$$p - \frac{\partial c(w, q^*)}{\partial q} \leq 0$$

and in interior solutions

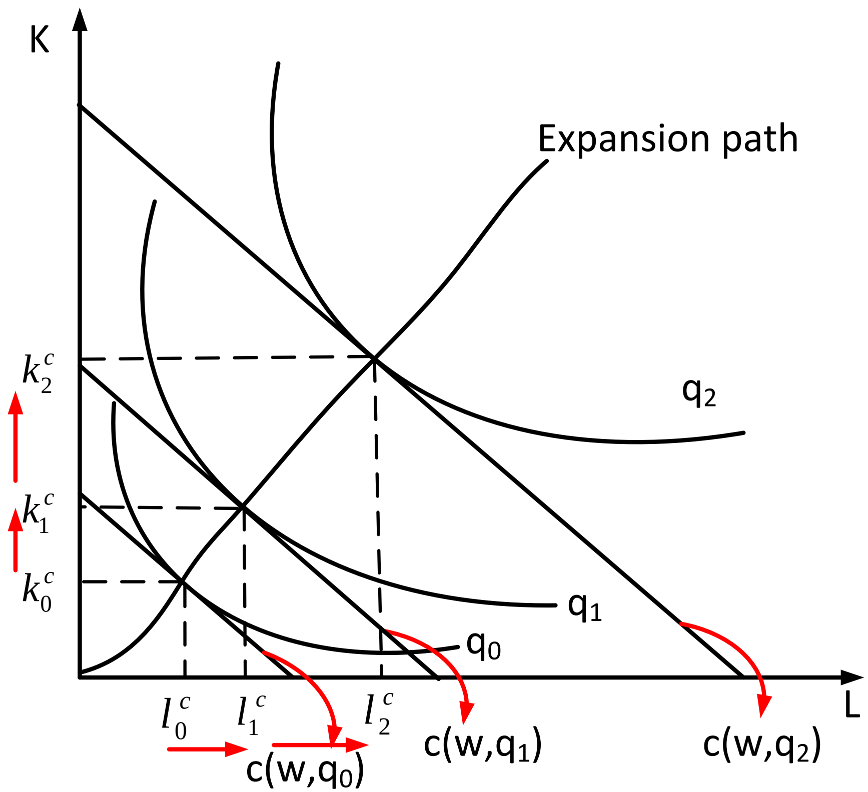
$$p - \frac{\partial c(w, q^*)}{\partial q} = 0$$

- That is, at the interior optimum q^*, price equals marginal cost, $\frac{\partial c(w,q^*)}{\partial q}$.

Firm's Expansion Path

- The ***expansion path*** is the locus of cost-minimizing tangencies. (Analogous to the *wealth expansion path* in consumer theory)
- The curve shows how inputs increase as output increases.
- Expansion path is positively sloped.
- Both k and l are *normal* goods, i.e.,

$$\frac{\partial k^c(w,q)}{\partial q} \geq 0, \frac{\partial l^c(w,q)}{\partial q} \geq 0$$



Firm's Expansion Path

- If the firm's expansion path is a ***straight line***:
  - All inputs must increase at a constant proportion as firm increases its output.
  - The firm's production function exhibits constant returns to scale and it is, hence, homothetic.
  - If the expansion path is straight and coincides with the 45-degree line, then the firm increases all inputs by the same proportion as output increases.
- The expansion path does not have to be a straight line.
  - The use of some inputs may increase faster than others as output expands
    - Depends on the shape of the isoquants.

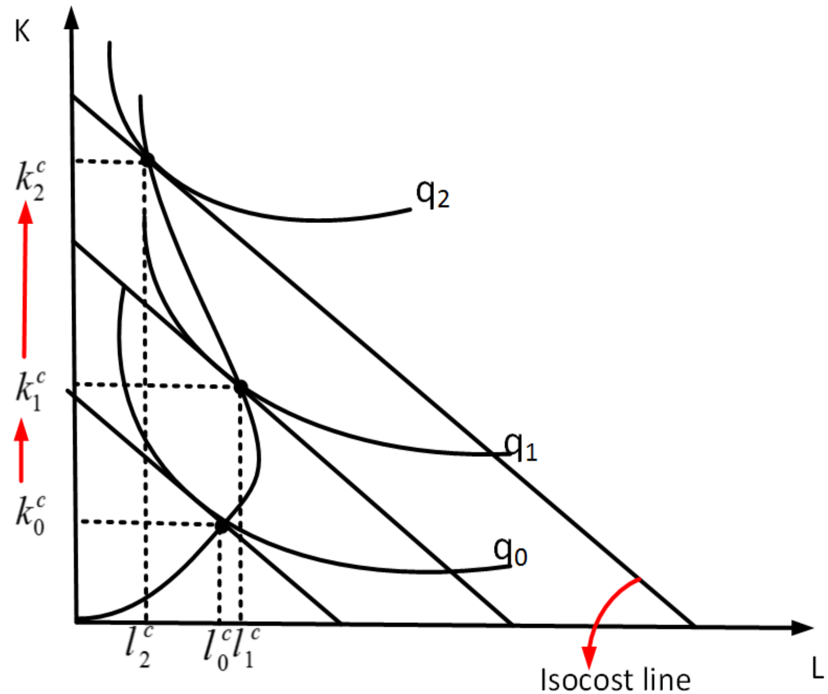
Firm's Expansion Path

- The expansion path does not have to be upward sloping.
  - If the use of an input falls as output expands, that input is an *inferior* input.
- k is *normal*

$$\frac{\partial k^c(w,q)}{\partial q} \geq 0$$

but l is inferior (at higher levels of output)

$$\frac{\partial l^c(w,q)}{\partial q} < 0$$



Firm's Expansion Path

- Are there inferior inputs out there?
  - We can identify inferior inputs if the list of inputs used by the firms is relatively disaggregated.
  - For instance, we can identify following categories: CEOs, executives, managers, accountants, secretaries, janitors, etc.
  - These inputs do not increase at a constant rate as the firm increases output (i.e., expansion path would not be a straight line for all increases in q).
  - After reaching a certain scale, the firm might buy a powerful computer with which accounting can be done using fewer accountants.

Cost and Supply: Single Output

- Let us assume a given vector of input prices $\bar{w} \gg 0$. Then, $c(\bar{w}, q)$ can be reduced to $C(q)$. Then, average and marginal costs are

$$AC(q) = \frac{C(q)}{q} \text{ and } MC = C'(q) = \frac{\partial C(q)}{\partial q}$$

- Hence, the FOCs of the PMP can be expressed as

$p \leq C'(q)$, and in interior solutions $p = C'(q)$

i.e., all output combinations such that $p = C'(q)$ are the (optimal) supply correspondence of the firm $q(p)$.

Cost and Supply: Single Output

- We showed that the cost function $c(w, q)$ is homogenous of degree 1 in input prices, w.
  - Can we extend this property to the AC and MC? Yes!
  - For average cost function,

$$AC(tw, q) = \frac{C(tw, q)}{q} = \frac{t \cdot C(w, q)}{q}$$
$$= t \cdot AC(tw, q)$$

Cost and Supply: Single Output

- For marginal cost function,

$$MC(tw,q) = \frac{\partial C(tw,q)}{\partial q} = \frac{t \cdot \partial C(w,q)}{\partial q}$$
$$= t \cdot MC(tw,q)$$

- Isn't this result violating Euler's theorem? No!
  - The above result states that $c(w,q)$ is homog(1) in inputs prices, and that $MC(w,q) = \frac{\partial C(w,q)}{\partial q}$ is also homog(1) in input prices.
  - Euler's theorem would say that: If $c(w,q)$ is homog(1) in inputs prices, then its derivate with respect to input prices, $\frac{\partial C(w,q)}{\partial w}$, must be homog(0).

Graphical Analysis of Total Cost

- With constant returns to scale, total costs are proportional to output.

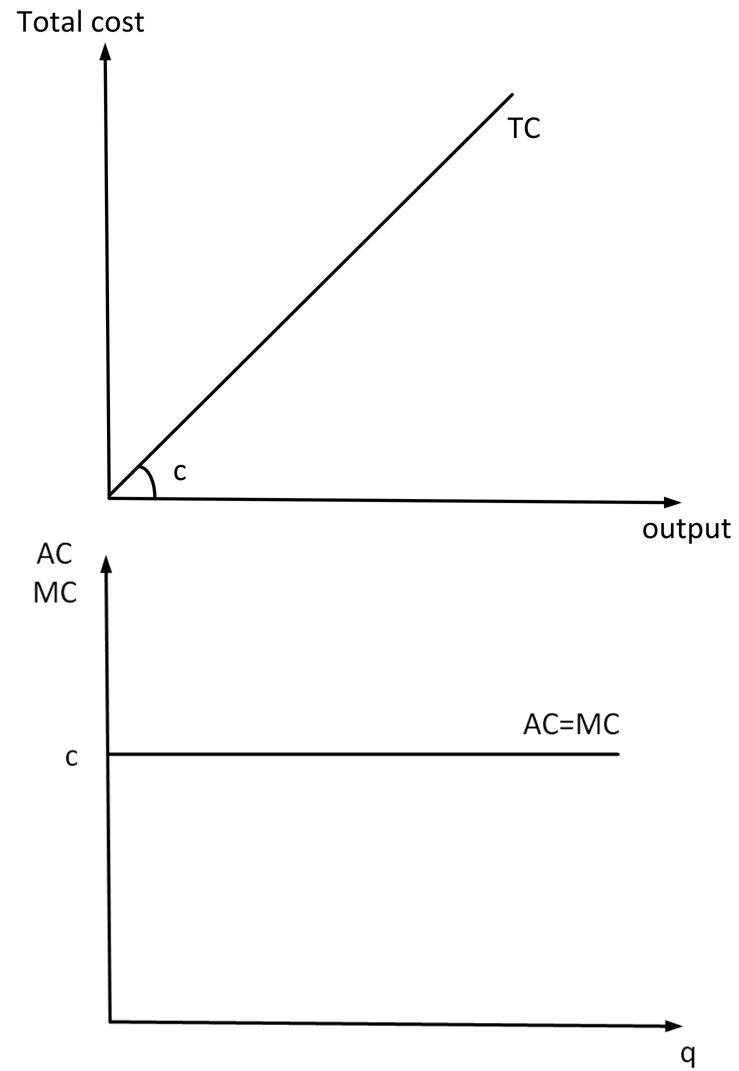
$$TC(q) = c \cdot q$$

- Hence,

$$AC(q) = \frac{TC(q)}{q} = c$$

$$MC(q) = \frac{\partial TC(q)}{\partial q} = c$$

$$\Longrightarrow AC(q) = MC(q)$$



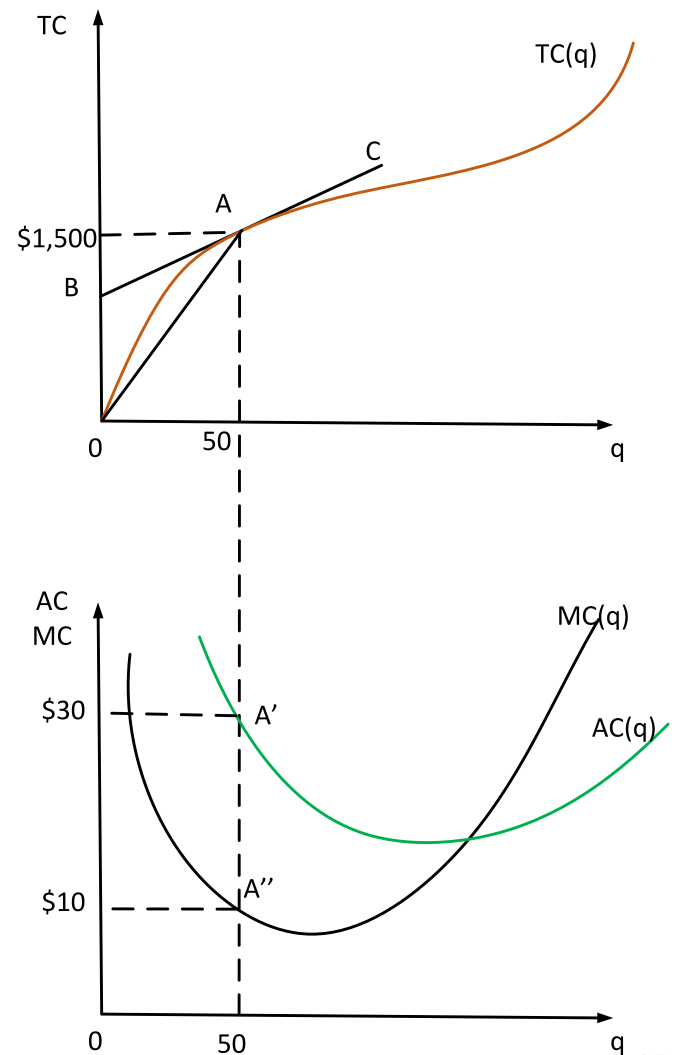
Cost and Supply: Single Output

- Suppose that TC starts out as concave and then becomes convex as output increases.
  - TC no longer exhibits constant returns to scale.
  - One possible explanation for this is that there is a third factor of production that is fixed as capital and labor usage expands (e.g., entrepreneurial skills).
  - TC begins rising rapidly after *diminishing returns* set in.

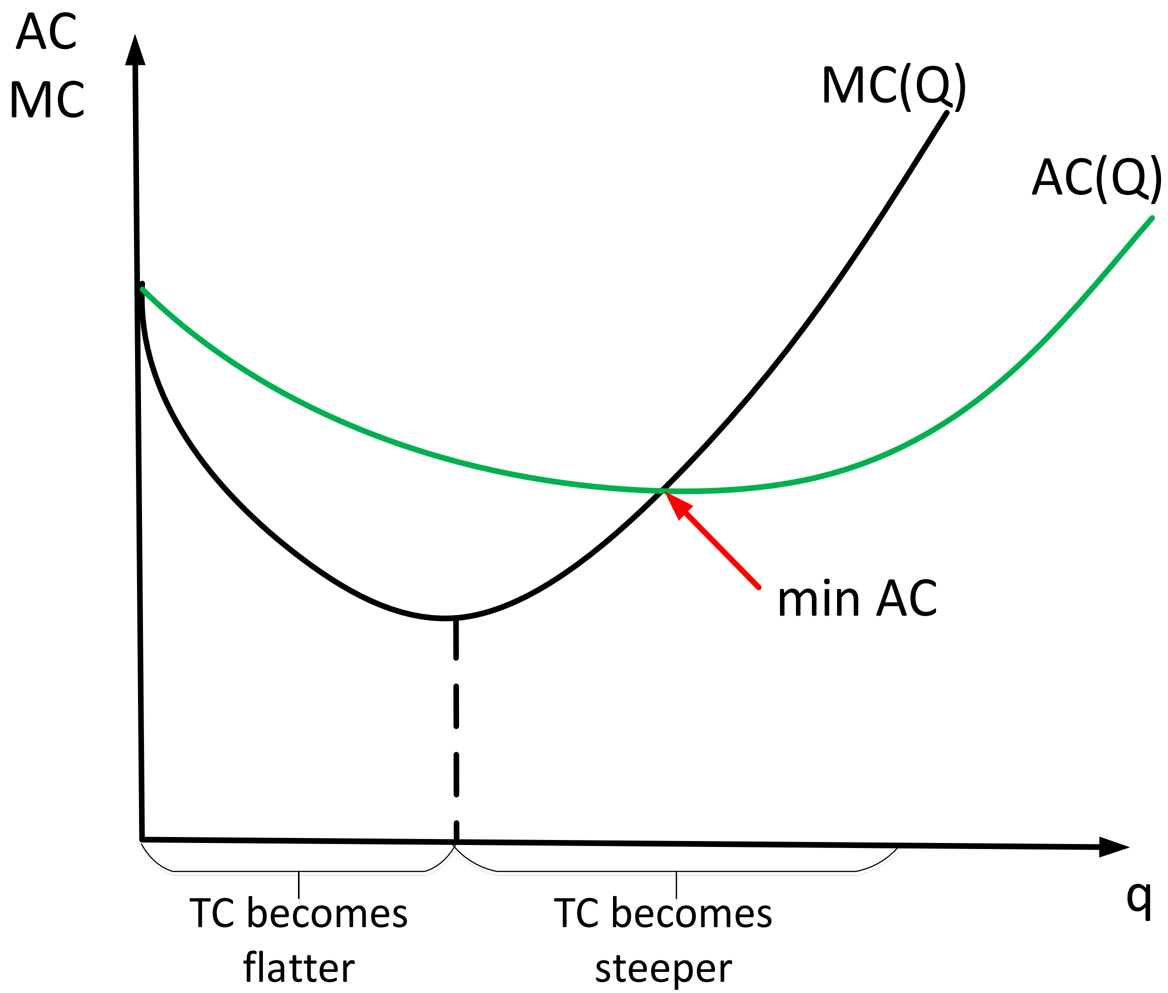
Cost and Supply: Single Output

- TC initially grows very rapidly, then becomes relatively flat, and for high production levels increases rapidly again.

- MC is the slope of the TC curve.



Cost and Supply: Single Output



Cost and Supply: Single Output

- ***Remark 1***: AC=MC at $q = 0$.
  - Note that we cannot compute
$$AC(0) = \frac{TC(0)}{0} = \frac{0}{0}$$
  - We can still apply l'Hopital's rule
$$\lim_{q \to 0} AC(q) = \lim_{q \to 0} \frac{TC(q)}{q} = \lim_{q \to 0} \frac{\frac{\partial TC(q)}{\partial q}}{\frac{\partial q}{\partial q}} = \lim_{q \to 0} MC(q)$$
  - Hence, AC=MC at $q = 0$, i.e., AC(0)=MC(0).

Cost and Supply: Single Output

- ***Remark 2***: When MC<AC, the AC curve decreases, and when MC>AC, the AC curve increases.
  - *Intuition*: using example of grades
  - If the new exam score raises your average grade, it must be that such new grade is better than your average grade thus far.
  - If, in contrast, the new exam score lowers your average grade, it must be that such new grade is than your average grade thus far.

Cost and Supply: Single Output

- ***Remark 3***: AC and MC curves cross (AC=MC) at exactly the minimum of the AC curve.
  - Let us first find the minimum of the AC curve

$$\frac{\partial AC(q)}{\partial q} = \frac{\partial\left(\frac{TC(q)}{q}\right)}{\partial q} = \frac{q\frac{\partial TC(q)}{\partial q} - TC(q)\cdot 1}{q^2}$$
$$= \frac{q \cdot MC(q) - TC(q)}{q^2} = 0$$

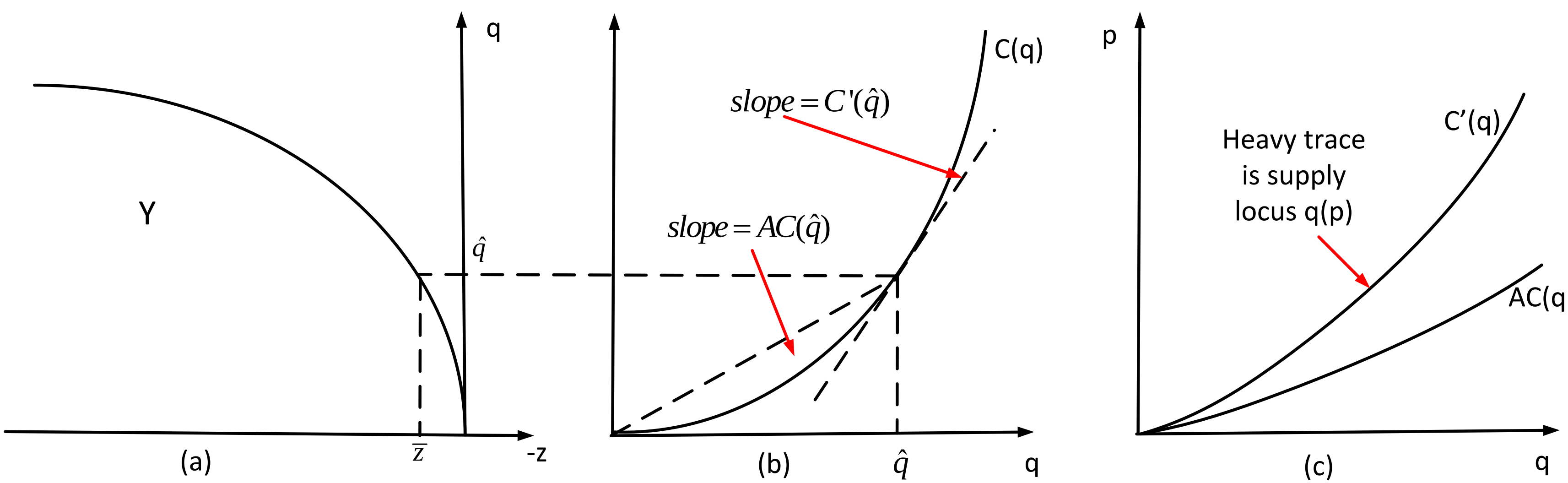
  - The output that minimizes AC must satisfy

$$q \cdot MC(q) - TC(q) = 0 \;\; \Longrightarrow MC(q) = \frac{TC(q)}{q} \leftarrow AC(q)$$

  - Hence, $MC = AC$ at the minimum of AC.

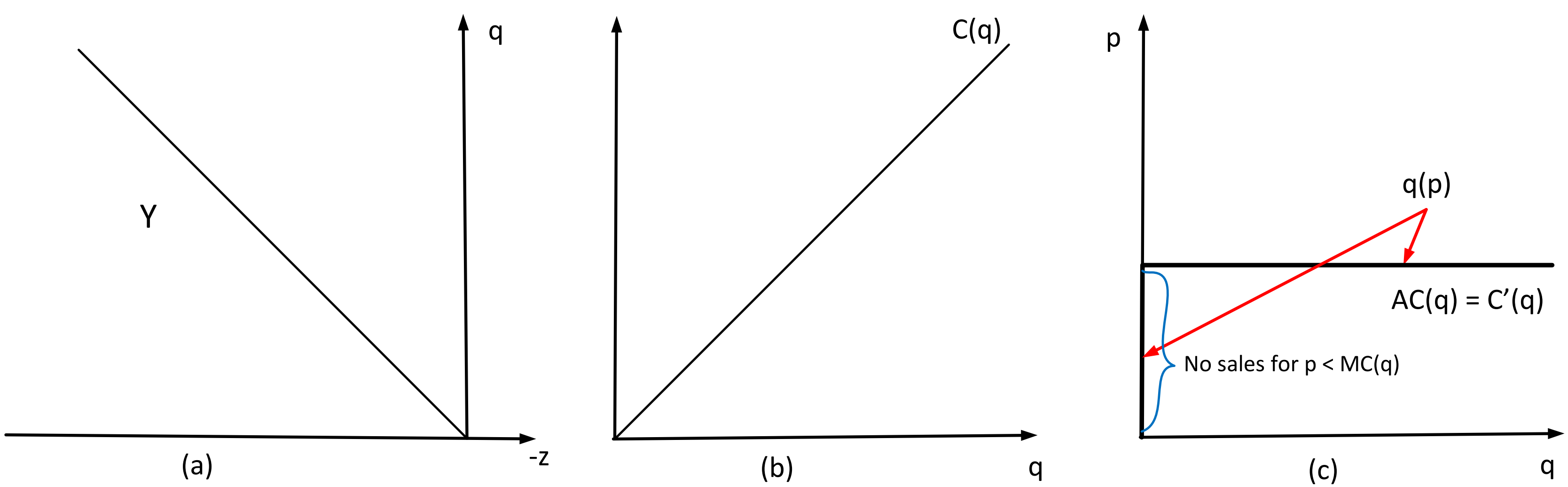
Cost and Supply: Single Output

- ***Decreasing returns to scale***:
  - an increase in the use of inputs produces a less-than-proportional increase in output.
    - production set is strictly convex
    - TC function is convex
    - MC and AC are increasing



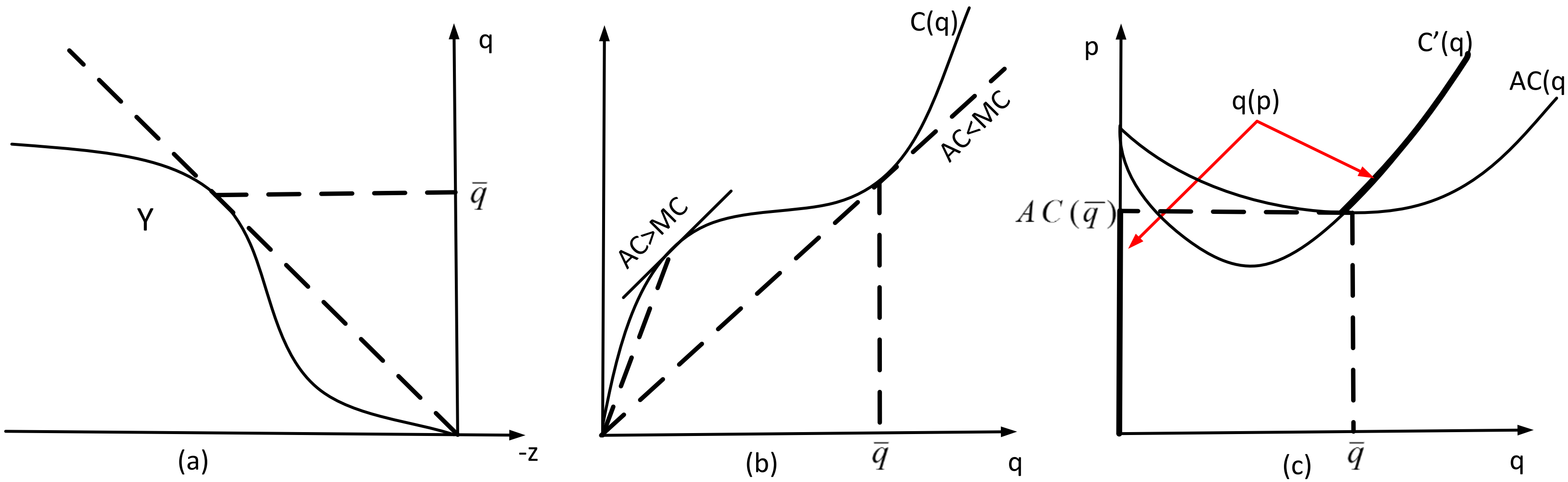
Cost and Supply: Single Output

- ***Constant returns to scale***:
  - an increase in input usage produces a proportional increase in output.
    - production set is weakly convex
    - linear TC function
    - constant AC and MC functions



Cost and Supply: Single Output

- ***Increasing returns to scale***:
  - an increase in input usage can lead to a more-than-proportional increase in output.
    - production set is non-convex
    - TC curve first increases, then becomes almost flat, and then increases rapidly again as output is increased further.



Cost and Supply: Single Output

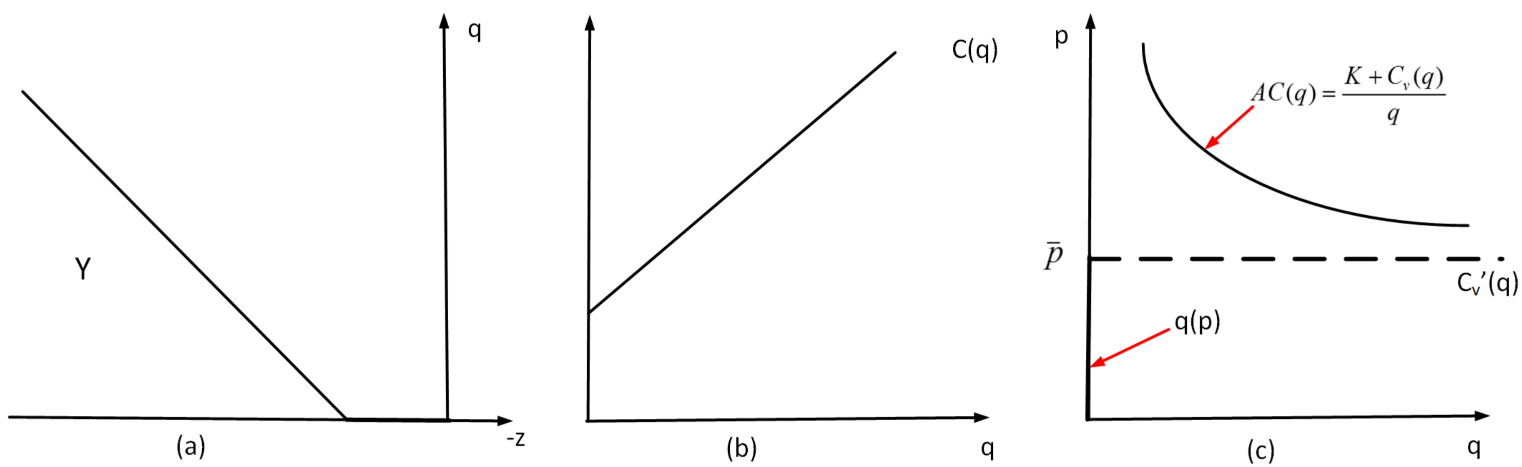
- Let us analyze the presence of *non-convexities* in the production set Y arising from:
  - Fixed set-up costs, K, that are non-sunk

$$C(q) = K + C_v(q)$$

    where $C_v(q)$ denotes variable costs
    - with strictly convex variable costs
    - with linear variable costs
  - Fixed set-up costs that are sunk
    - Cost function is convex, and hence FOCs are sufficient

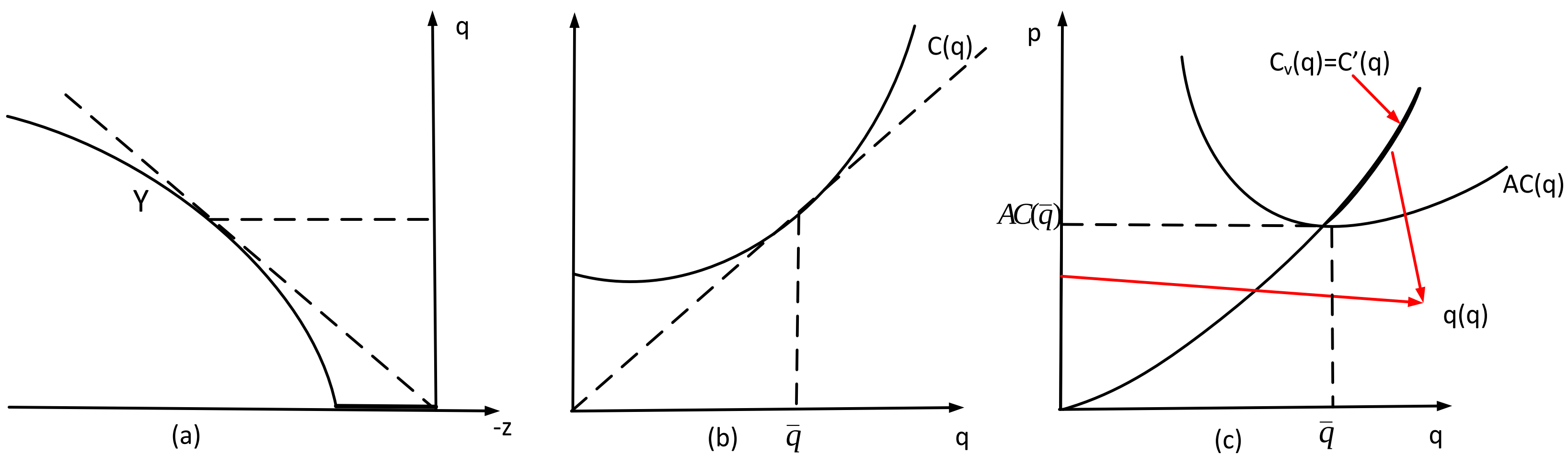
Cost and Supply: Single Output

- ***CRS technology and fixed (non-sunk) costs***:
  - If $q = 0$, then $C(q) = 0$, i.e., firm can recover K if it shuts down its operation.
  - MC is constant: $MC = C'(q) = C_v'(q) = c$
  - AC lies above MC: $AC(q) = \frac{C(q)}{q} = \frac{K}{q} + \frac{C_v(q)}{q} = \frac{K}{q} + c$



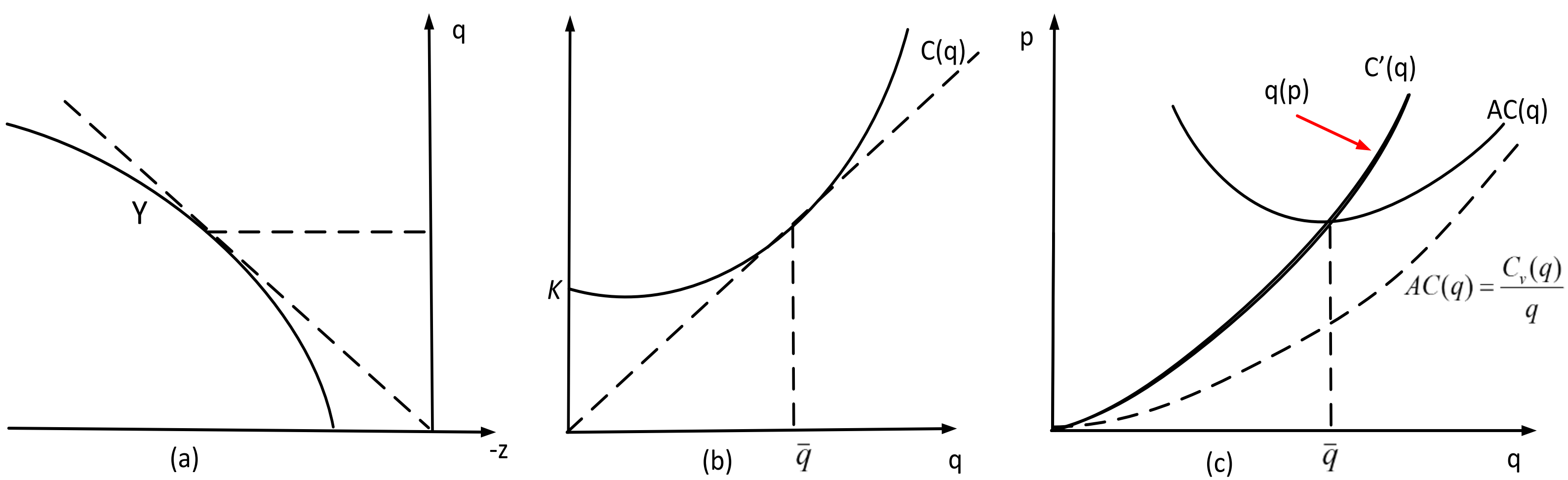
Cost and Supply: Single Output

- ***DRS technology and fixed (non-sunk) costs***:
  - MC is positive and increasing in q, and hence the slope of the TC curve increases in q.
  - in the decreasing portion of the AC curve, FC is spread over larger q.
  - in the increasing portion of the AC curve, larger average VC offsets the lower average FC and, hence, total average cost increases.



Cost and Supply: Single Output

- ***DRS technology and sunk costs***:
  - TC curve originates at K, given that the firm must incur fixed sunk cost K even if it chooses $q = 0$.
  - supply locus considers the entire MC curve and not only q for which MC>AC.

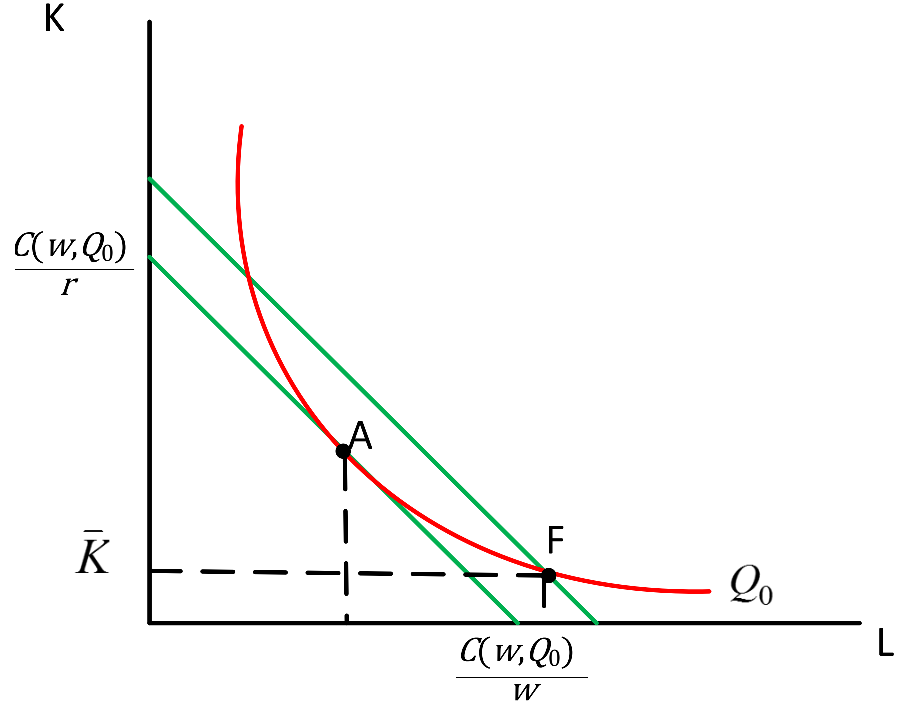


Short-Run Total Cost

- In the short run, the firm generally incurs higher costs than in the long run
  - The firm does not have the flexibility of input choice (fixed inputs).
  - To vary its output in the short-run, the firm must use non-optimal input combinations
  - The $MRTS$ will not be equal to the ratio of input prices.

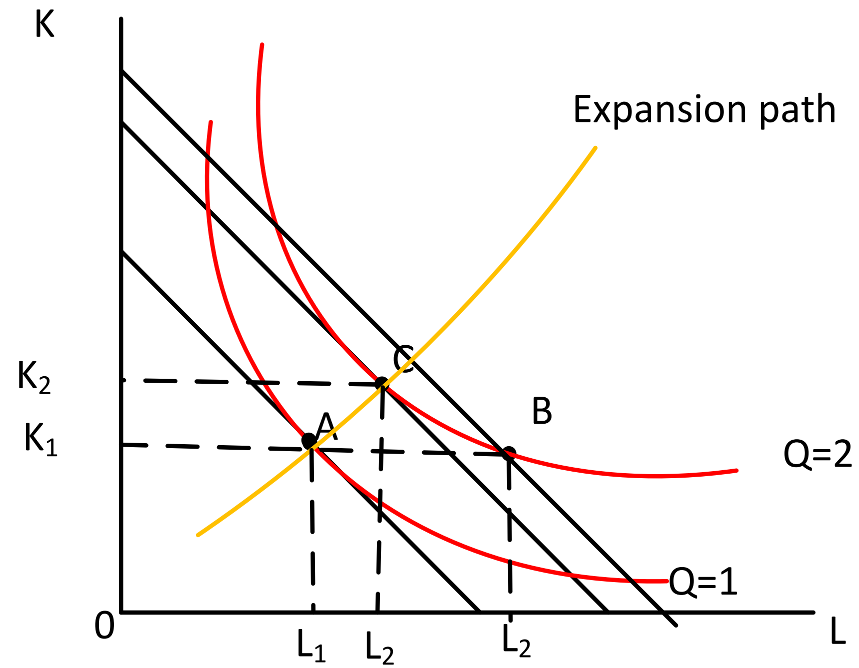
Short-Run vs Long-Run Total Cost

- In the short-run
  - capital is fixed at $\bar{K}$
  - the firm cannot equate $MRTS$ with the ratio of input prices.
- In the long-run
  - Firm can choose input vector A, which is a cost-minimizing input combination.



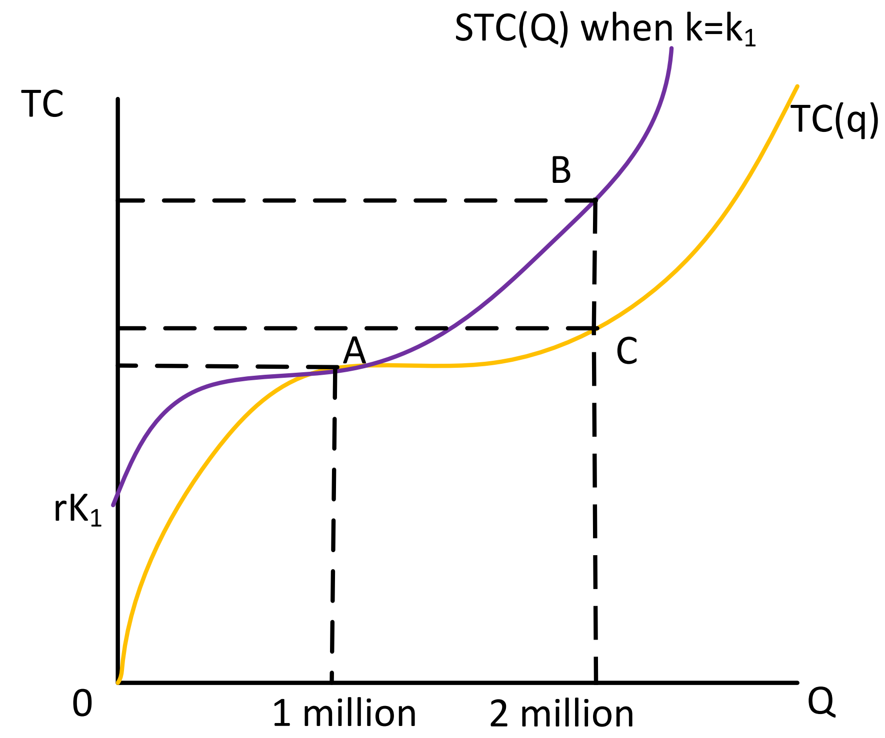
Short-Run vs Long-Run Total Cost

- $q = 1$ million units
  - Firm chooses (k_1, l_1) both in the long run and in the short run when $k = k_1$.
- $q = 2$ million units
  - Short-run (point B):
    - $k = k_1$ does not allow the firm to minimize costs.
  - Long-run (point C):
    - firm can choose cost-minimizing input combination.



Short-Run vs Long-Run Total Cost

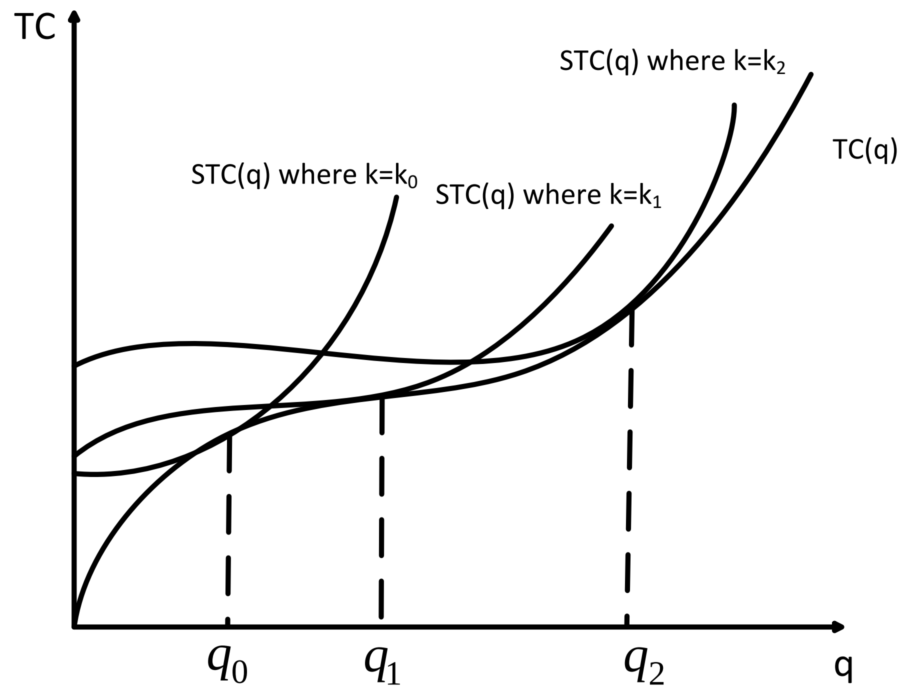
- The difference between long-run, $TC(q)$, and short-run, $STC(q)$, total costs when capital is fixed at $k = k_1$.



Short-Run vs Long-Run Total Cost

- The long-run total cost curve $TC(q)$ can be derived by varying the level of k.

- Short-run total cost curve $STC(q)$ lies above long-run total cost $TC(q)$.



Short-Run vs Long-Run Total Cost

- Summary:
  - In the long run, the firm can modify the values of all inputs.
  - In the short run, in contrast, the firm can only modify some inputs (e.g., labor, but not capital).

Short-Run vs Long-Run Total Cost

- ***Example***: Short- and long-run curves
  - In the long run,
    $$C(q) = \bar{w}_1 z_1 + \bar{w}_2 z_2$$
    where both input 1 and 2 are variable.
  - In the short run, input 2 is fixed at $\bar{z}_2$, and thus
    $$C(q|\bar{z}_2) = \bar{w}_1 z_1 + \bar{w}_2 \bar{z}_2$$
    - This implies that the only input that the firm can modify is input 1.
    - The firm chooses z_1 such that production reaches output level q, i.e., $f(z_1, \bar{z}_2) = q$.

Short-Run vs Long-Run Total Cost

- ***Example*** (continued):
  - When the demand for input 2 is at its *long-run value*, i.e., $z_2(w, q)$, then

    $$C(q) = C(q|z_2(w, q)) \text{ for every } q$$

    and also

    $$C'(q) = C'(q|z_2(w, q)) \text{ for every } q$$

    i.e., values and slopes of long- and short-run cost functions coincide.
  - Long- and short-run curves are tangent at $z_2(w, q)$.

Short-Run vs Long-Run Total Cost

- ***Example*** (continued):
  - Since

$$C(q) \leq C(q|z_2) \text{ for any given } z_2,$$

then the long-run cost curve $C(q)$ is the *lower envelope* of the short-run cost curves, $C(q|z_2)$.

Aggregation in Production

Aggregation in Production

- Let us analyze under which conditions the "law of supply" holds at the aggregate level.
- An aggregate production function maps aggregate inputs into aggregate outputs
  - In other words, it describes the maximum level of output that can be obtained if the inputs are efficiently used in the production process.

Aggregation in Production

- Consider J firms, with production sets $Y_1, Y_2, \dots, Y_J$.
- Each Y_j is non-empty, closed, and satisfies the free disposal property.
- Assume also that every supply correspondence $y_j(p)$ is single valued, and differentiable in prices, $p \gg 0$.
- Define the aggregate supply correspondence as the sum of the individual supply correspondences

$$y(p) = \sum_{j=1}^{J} y_j(p) = \left\{ y \in \mathbb{R}^L : y = \sum_{j=1}^{J} y_j(p) \right\}$$

where $y_j \in y_j(p)$ for $j = 1,2, \dots, J$.

Aggregation in Production

- The ***law of supply*** is satisfied at the aggregate level.
- Two ways to check it:
  1) Using the derivative of every firm's supply correspondence with respect to prices, $D_p y_j(p)$.
     - $D_p y_j(p)$ is a symmetric positive semidefinite matrix, for every firm j.
     - Since this property is preserved under addition, then $D_p y(p)$ must also define a symmetric positive semidefinite matrix.

Aggregation in Production

2) Using a revealed preference argument.
   - For every firm j,
     $$[p - p'] \cdot \left[y_j(p) - y_j(p')\right] \geq 0$$
   - Adding over j,
     $$[p - p'] \cdot [y(p) - y(p')] \geq 0$$
   - This implies that market prices and aggregate supply move in the same direction
     - the law of supply holds at the aggregate level!

Aggregation in Production

- Is there a "***representative producer***"?
  - Let Y be the aggregate production set,
    $$Y = Y_1 + Y_2 + \ldots + Y_j = \left\{ y \in \mathbb{R}^L \colon y = \sum_{j=1}^{J} y_j \right\}$$
    for some $y_j \in Y_j$ and $j = 1,2, \ldots, J$.
  - Note that $y = \sum_{j=1}^{J} y_j$, where every y_j is just a feasible production plan of firm j, but not necessarily firm j's supply correspondence $y_j(p)$.
  - Let $\pi^*(p)$ be the profit function for the aggregate production set Y.
  - Let $y^*(p)$ be the supply correspondence for the aggregate production set Y.

Aggregation in Production

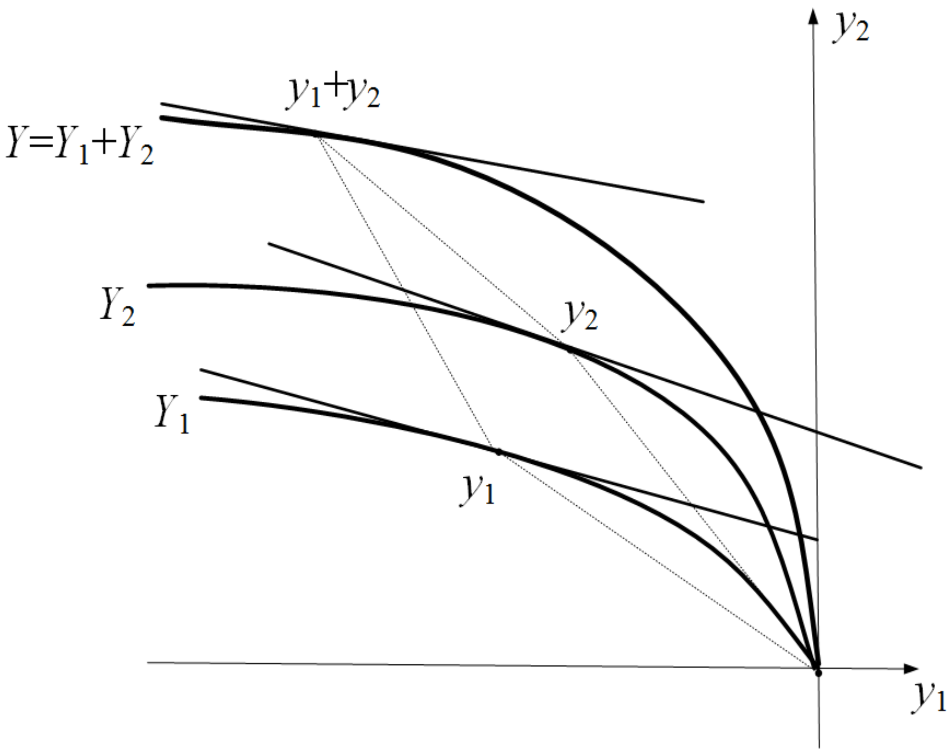
- Is there a "***representative producer***"?
  - Then, there exists a representative producer:
    - Producing an aggregate supply $y^*(p)$ that exactly coincides with the sum $\sum_{j=1}^{J} y_j\,(p)$; and
    - Obtaining aggregate profits $\pi^*(p)$ that exactly coincide with the sum $\sum_{j=1}^{J} \pi_j\,(p)$.
  - *Intuition*: The aggregate profit obtained by each firm maximizing its profits separately (taking prices as given) is the same as that which would be obtained if all firms were to coordinate their actions (i.e., y_j's) in a joint PMP.

Aggregation in Production

- Is there a "***representative producer***"?
  - It is a "decentralization" result: to find the solution of the joint PMP for given prices p, it is enough to "let each individual firm maximize its own profits" and add the solutions of their individual PMPs.
  - *Key*: price taking assumption
    - This result does not hold if firms have market power.
    - Example: oligopoly markets where firms compete in quantities (a la Cournot).

Aggregation in Production

- Firm 1 chooses y_1 given p and Y_1.
- Firm 2 chooses y_2 given p and Y_2.
- Jointly, the two firms would be selecting $y_1 + y_2$.
- The aggregate supply correspondence $y_1 + y_2$ coincides with the supply correspondence that a single firm would select given p and $Y = y_1 + y_2$.



Efficient Production

Efficient Production

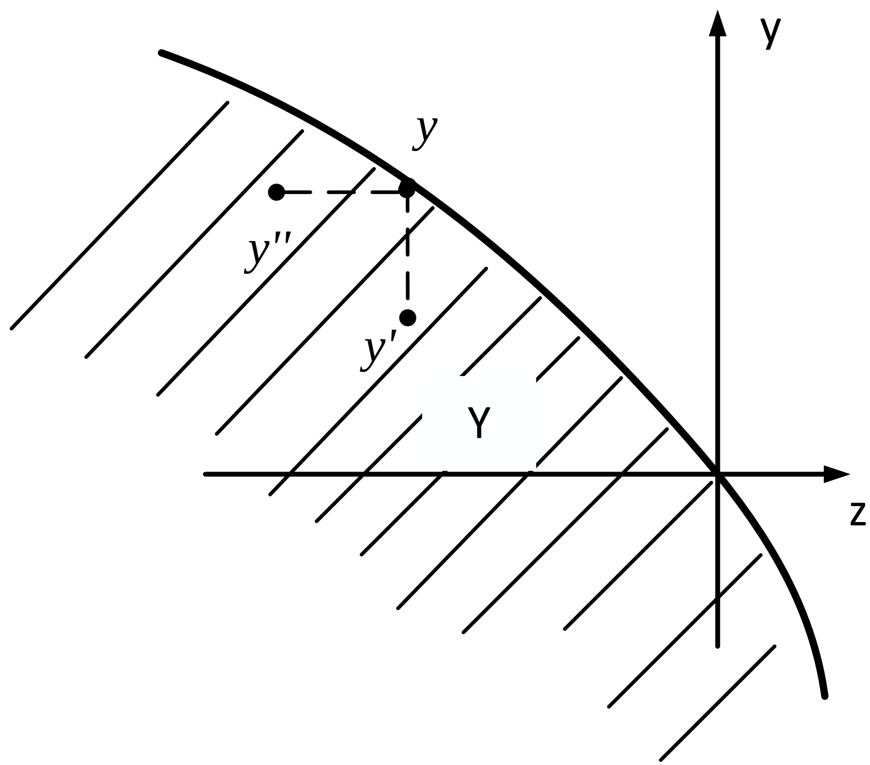
- ***Efficient production vector***: a production vector $y \in Y$ is efficient if there is no other $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.
  - That is, y is efficient if there is no other feasible production vector y' producing more output with the same amount of inputs, or alternatively, producing the same output with fewer inputs.

$$y \text{ is efficient } \Rightarrow \ y \text{ is on the boundary of } Y$$

$$y \text{ is efficient } \nLeftarrow \ y \text{ is on the boundary of } Y$$

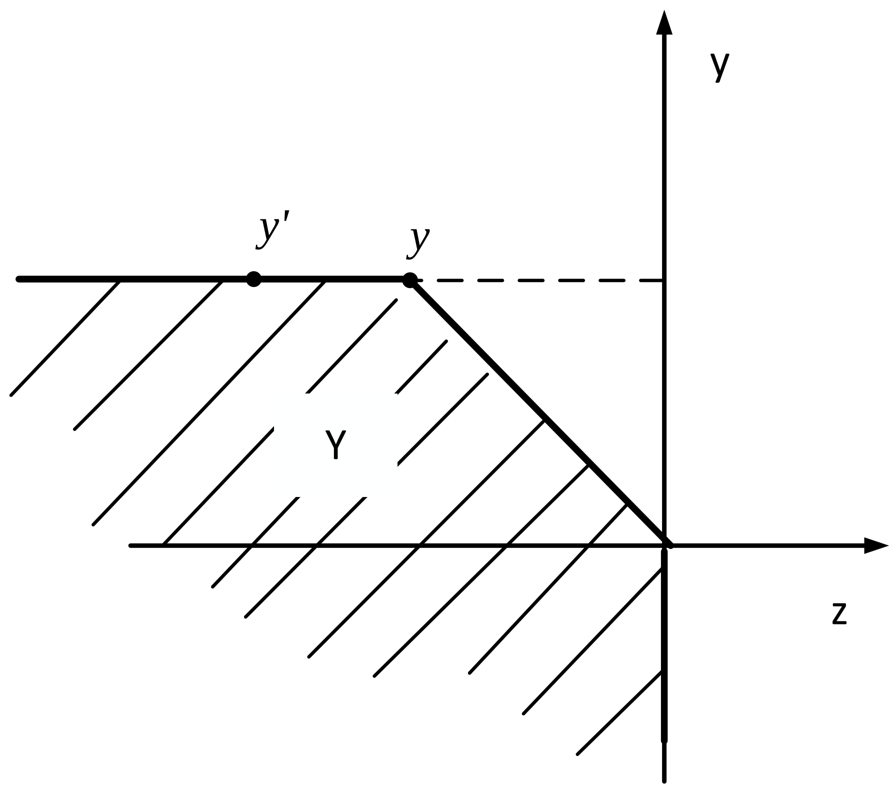
Efficient Production

- y'' produces the same output as y, but uses more inputs.
- y' uses the same inputs as y, but produces less output.
- y'' and y' are inefficient.
- y is efficient $\Rightarrow y$ lies on the frontier of the production set Y.



Efficient Production

- y is efficient
- y' is inefficient
  - it produces the same output as y, but uses more inputs.
- Hence, y' lies on the frontier of the production set $Y \nRightarrow y'$ is efficient.



Efficient Production: 1st FTWE

- ***1st Fundamental Theorem of Welfare Economics (FTWE)***: If $y \in Y$ is profit-maximizing for some price vector $p \gg 0$, then y must be efficient.

  Proof: Let us prove the 1st FTWE by contradiction. Suppose that $y \in Y$ is profit-maximizing

$$p \cdot y \geq p \cdot y' \text{ for all } y' \in Y$$

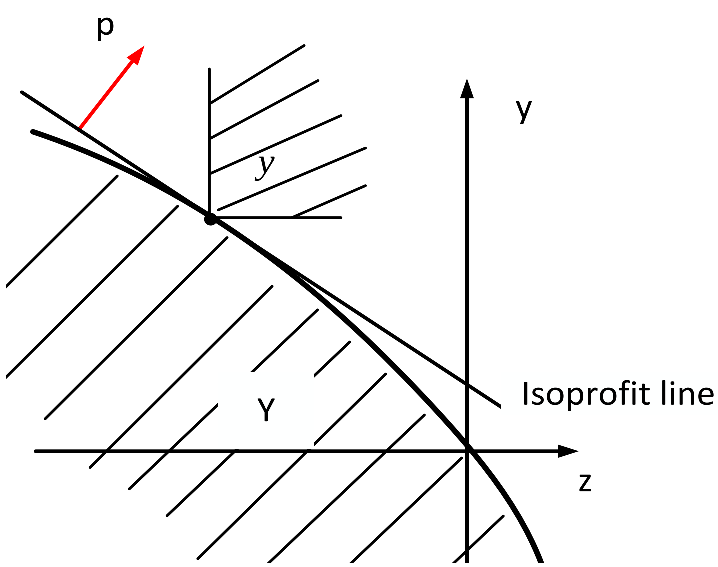
  but y is not efficient. That is, there is a $y' \in Y$ such that $y' \geq y$. If we multiply both sides of $y' \geq y$ by p, we obtain

$$p \cdot y' \geq p \cdot y, \text{ since } p \gg 0$$

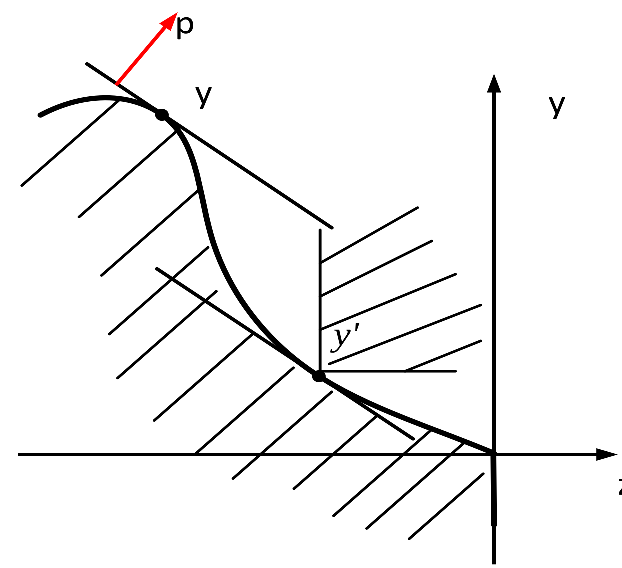
  But then, y cannot be profit-maximizing. Contradiction!

Efficient Production: 1st FTWE

- For the result in 1st FTWE, we do NOT need the production set Y to be convex.
  - y is profit-maximizing $\Rightarrow$ y lies on a tangency point



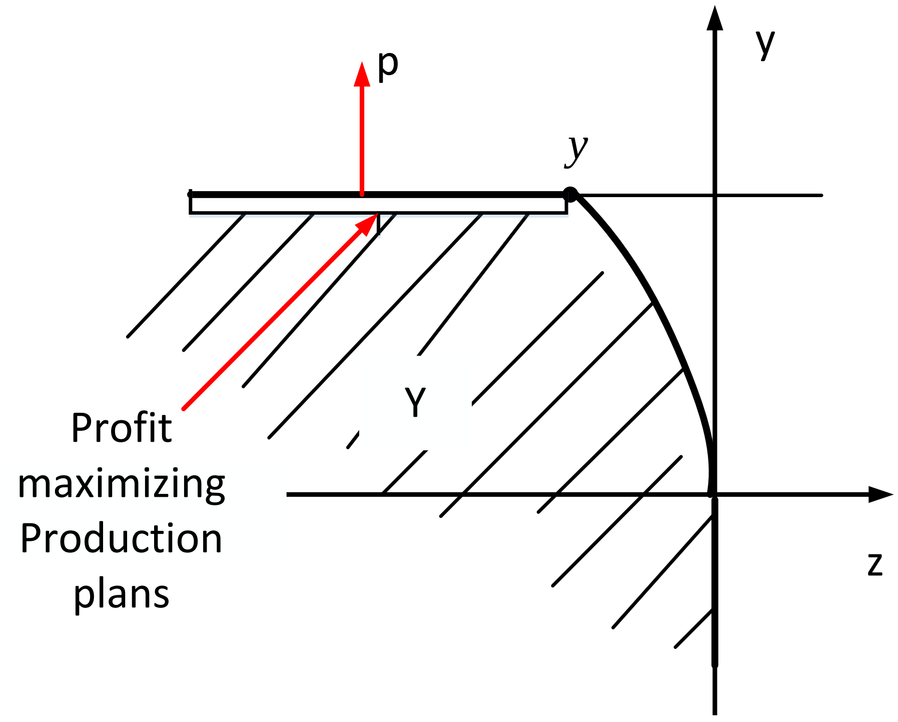
convex production set



non-convex production set

Efficient Production: 1^{st} FTWE

- **Note**: the assumption $p \gg 0$ cannot be relaxed to $p \geq 0$.
  - Take a production set Y with an upper flat surface.
  - Any production plan in the flat segment of Y can be profit-maximizing if prices are $p = (0,1)$.
  - But only y is efficient.
  - Other profit-maximizing production plans to the left of y are NOT efficient.
  - Hence, in order to apply 1^{st} FTWE we need $p \gg 0$.

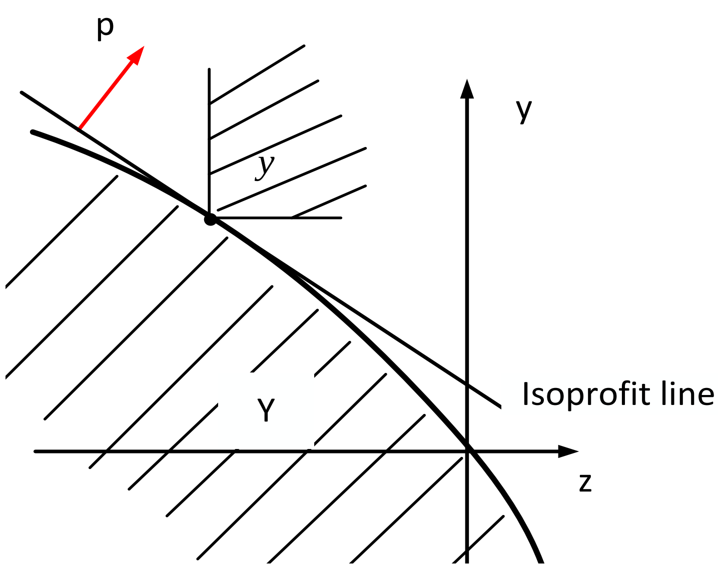


Efficient Production: 2nd FTWE

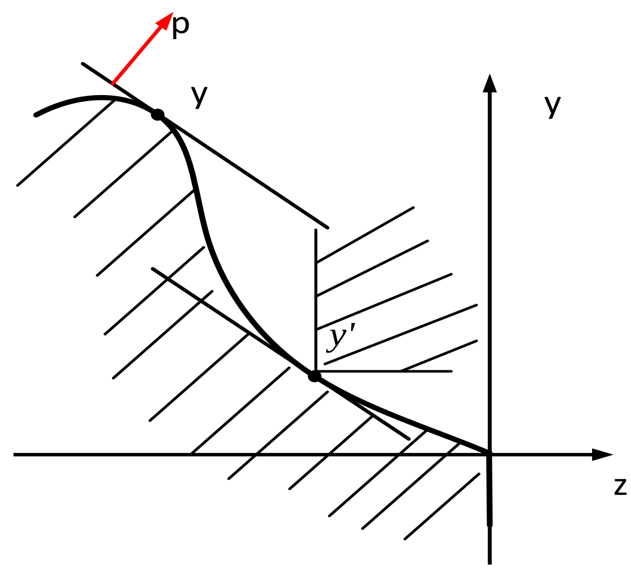
- The 2nd FTWE states the converse of the 1st FTWE:
  - If a production plan y is efficient, then it must be profit-maximizing.
- Note that, while it is true for convex production sets, it cannot be true if Y is non-convex.

Efficient Production: 2nd FTWE

- The 2nd FTWE is restricted to convex production sets.
- For non-convex production set: If y is efficient $\not\Rightarrow$ y is profit-maximizing



convex production set



non-convex production set

Efficient Production: 2nd FTWE

- ***2nd FTWE***: If production set Y is convex, then every efficient production plan $y \in Y$ is profit-maximizing production plan, for some non-zero price vector $p \geq 0$.

Proof:

1) Take an efficient production plan, such as y on the boundary of Y. Define the set of production plans that are *strictly more efficient* than y

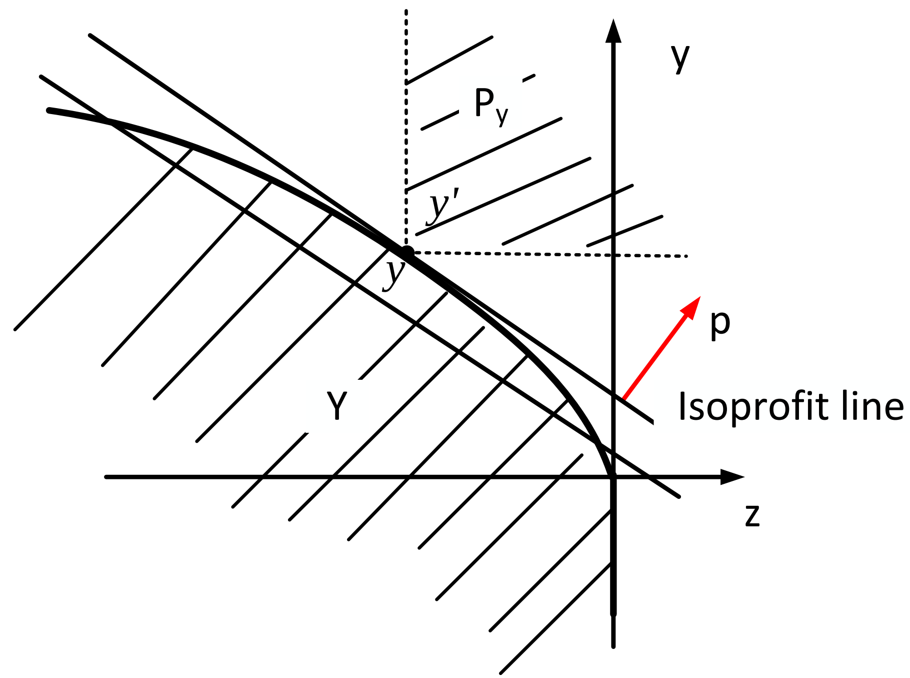
$$P_y = \{y' \in \mathbb{R}^L \colon y' \gg y\}$$

2) Note that $Y \cap P_y \neq \emptyset$ and P_y is convex set.

Efficient Production: 2^{nd} FTWE

Proof (continued):

3) From the Separating Hyperplane Theorem, there is some $p \neq 0$ such that $p \cdot y' \geq p \cdot y''$, for $y' \in P_y$ and $y'' \in Y$.

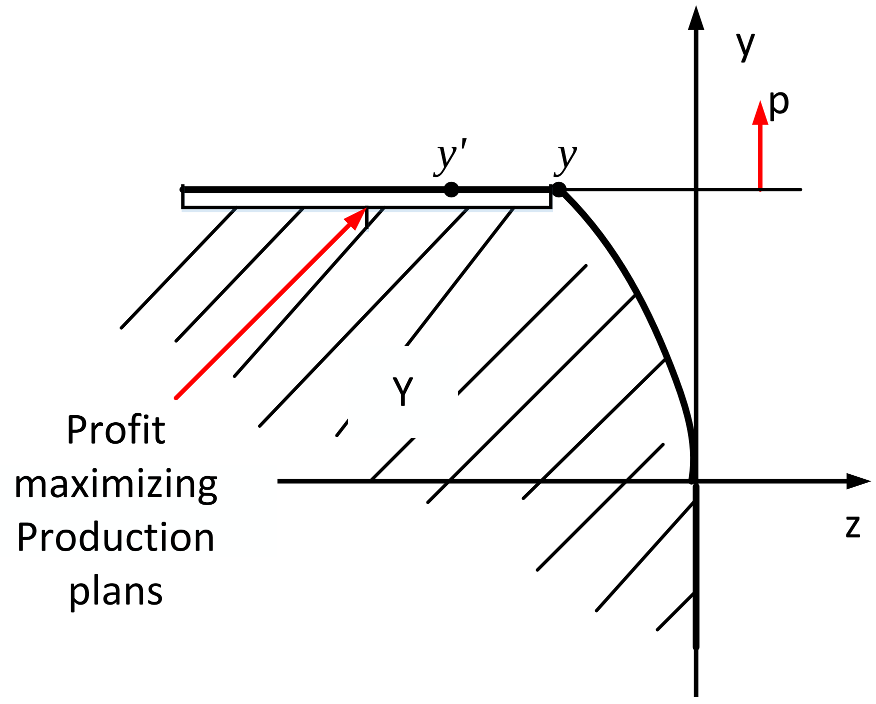


4) Since y' can be made arbitrarily close to y, we can have $p \cdot y \geq p \cdot y''$ for $y'' \in Y$.

5) Hence, the efficient production plan y must be profit-maximizing.

Efficient Production: 2nd FTWE

- **Note**: we are not imposing $p \gg 0$, but $p \geq 0$.
  - We just assume that the price vector is not zero for every component, i.e., $p \neq (0,0, \ldots, 0)$.
  - Hence, the slope of the isoprofit line can be zero.
  - Both y and y' are profit-maximizing, but only y is efficient.



Efficient Production: 2nd FTWE

- **Note**: the 2nd FTWE does not allow for input prices to be negative.
  - Consider the case in which the price of input l is negative, $p_l < 0$.
  - Then, we would have $p \cdot y' < p \cdot y$ for some production plan y' that is more efficient than y, i.e., $y' \gg y$, with $y'_l - y_l$ being sufficiently large.
  - This implies that the firm is essentially "paid" for using further amounts of input l.
  - For this reason, we assume $p \geq 0$.