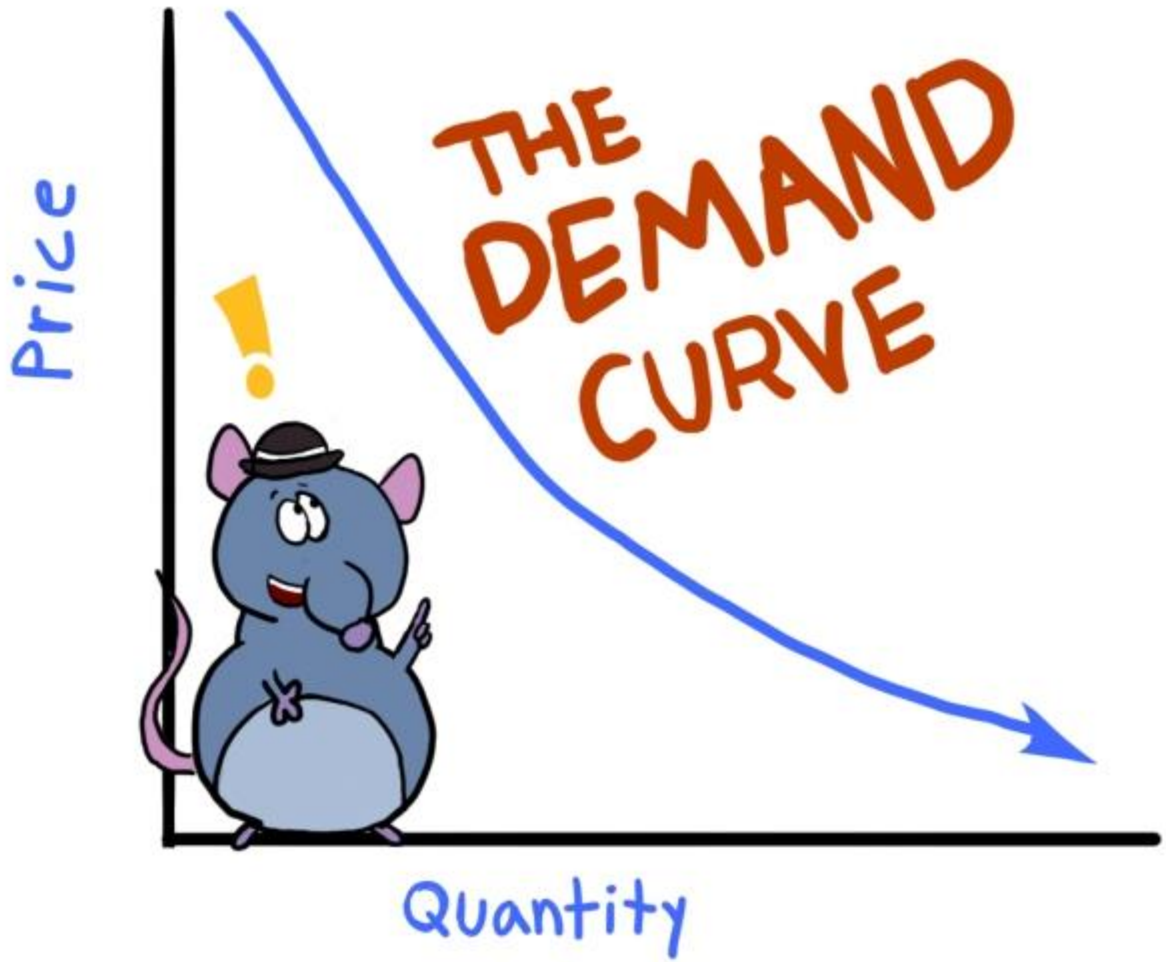
Demand Theory



Utility Maximization Problem

Utility Maximization Problem

- Consumer maximizes his utility level by selecting a bundle x (where x can be a vector) subject to his budget constraint:

$$\max_{x \geq 0} u(x)$$
$$\text{s.t.} \quad p \cdot x \leq w$$

- ***Weierstrass Theorem***: for optimization problems defined on the reals, if the objective function is continuous and constraints define a closed and bounded set, then the solution to such optimization problem exists.

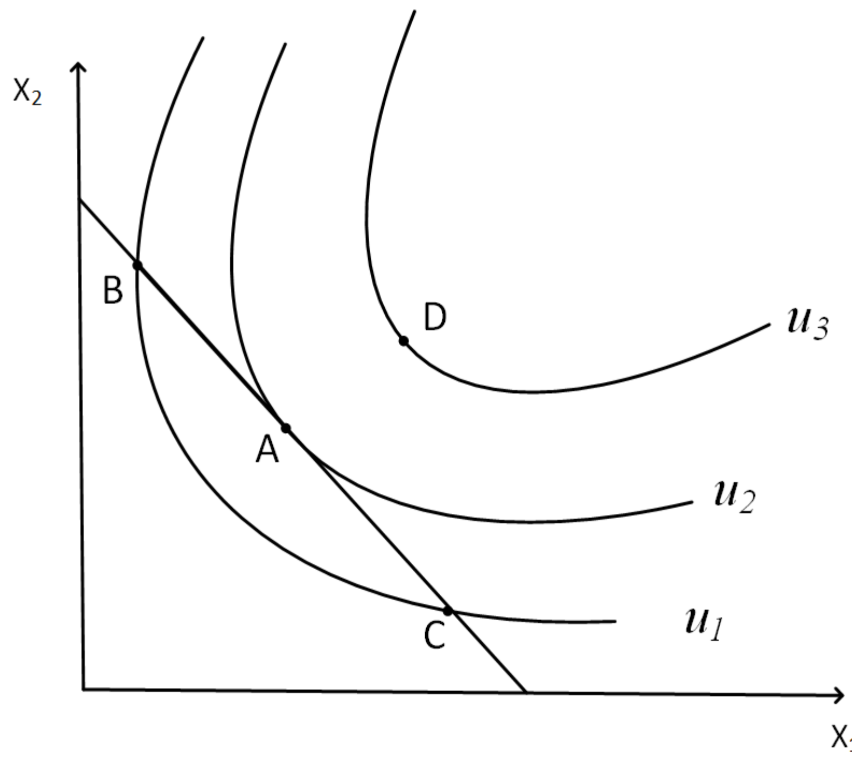
Utility Maximization Problem

- ***Existence***: if $p \gg 0$ and $w > 0$ (i.e., if $B_{p,w}$ is closed and bounded), and if $u(\cdot)$ is continuous, then there exists at least one solution to the UMP.
  – If, in addition, preferences are strictly convex, then the solution to the UMP is unique.

- We denote the solution of the UMP as the **argmax** of the UMP (the argument, x, that solves the optimization problem), and we denote it as $x(p, w)$.
  – $x(p, w)$ is the ***Walrasian demand*** correspondence, which specifies a demand of every good in $\mathbb{R}_{+}^{L}$ for every possible price vector, p, and every possible wealth level, w.

Utility Maximization Problem

- Walrasian demand $x(p, w)$ at bundle *A* is optimal, as the consumer reaches a utility level of u_2 by exhausting all his wealth.
- Bundles *B* and *C* are not optimal, despite exhausting the consumer's wealth. They yield a lower utility level u_1, where $u_1 < u_2$.
- Bundle *D* is unaffordable and, hence, it cannot be the argmax of the UMP given a wealth level of w.



Properties of Walrasian Demand

- If the utility function is continuous and preferences satisfy LNS over the consumption set $X = \mathbb{R}^L_+$, then the Walrasian demand $x(p, w)$ satisfies:

  1) Homogeneity of degree zero:

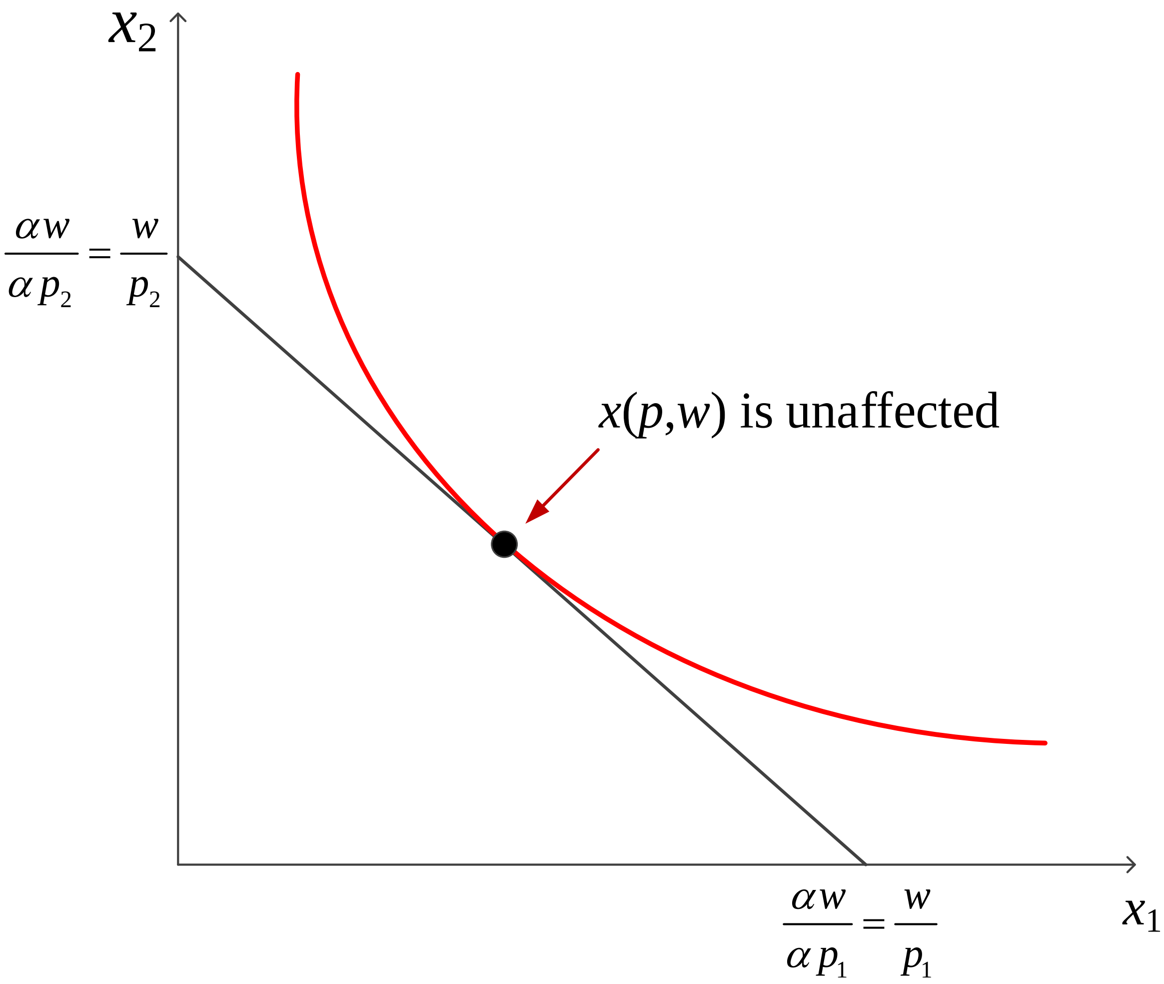
  $$x(p, w) = x(\alpha p, \alpha w) \text{ for all } p, w, \text{ and for all } \alpha > 0$$

  That is, the budget set is unchanged!

  $$\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\} = \{x \in \mathbb{R}^L_+ : \alpha p \cdot x \leq \alpha w\}$$

  Note that we don't need any assumption on the preference relation to show this. We only rely on the budget set being affected.

Properties of Walrasian Demand



– Note that the preference relation can be linear, and homog(0) would still hold.

Properties of Walrasian Demand

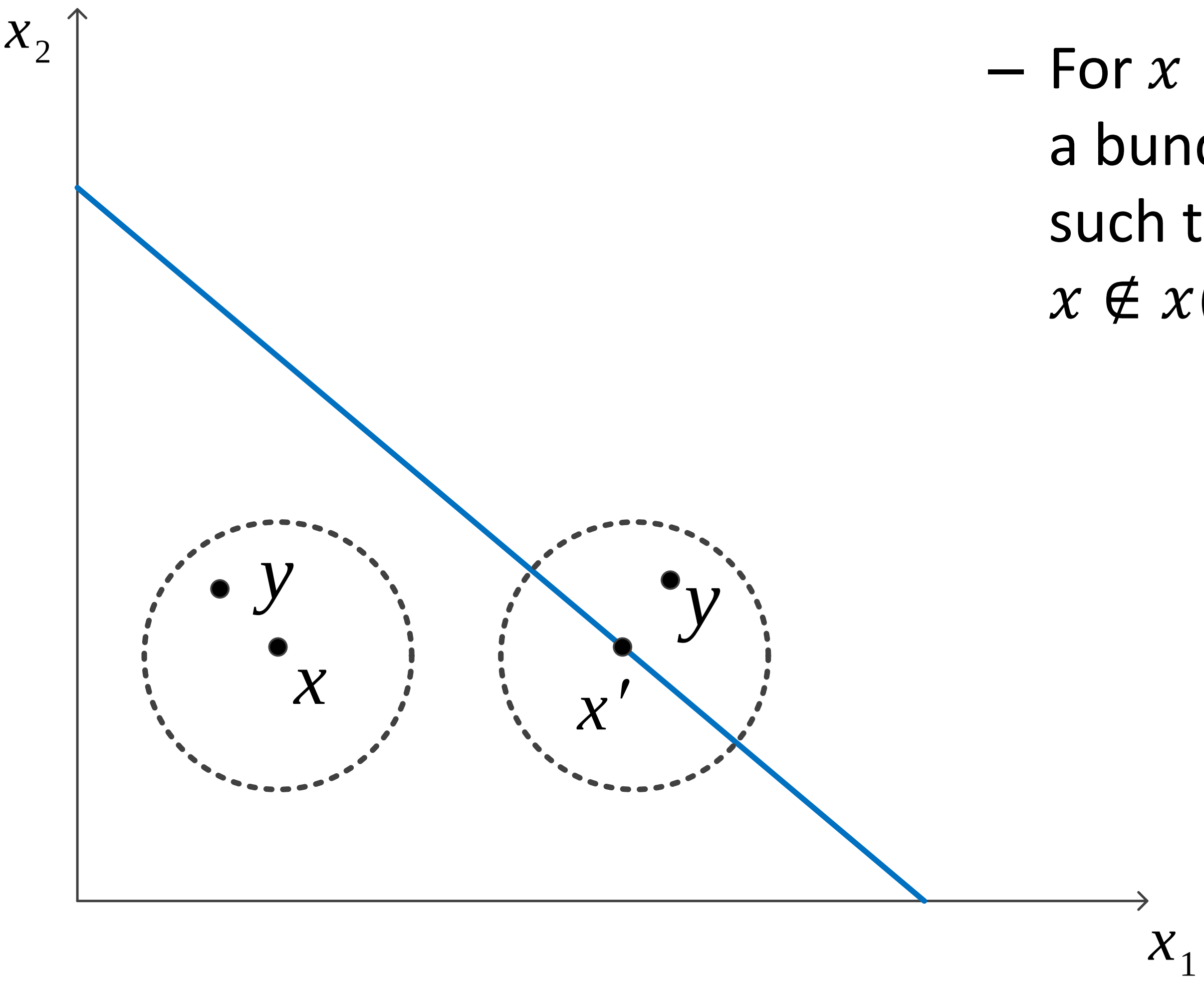
2) Walras' Law:

$$p \cdot x = w \quad \text{for all } x = x(p, w)$$

It follows from LNS: if the consumer selects a Walrasian demand $x \in x(p, w)$, where $p \cdot x < w$, then it means we can still find other bundle y, which is ε–close to x, where consumer can improve his utility level.

If the bundle the consumer chooses lies on the budget line, i.e., $p \cdot x' = w$, we could then identify bundles that are *strictly* preferred to x', but these bundles would be unaffordable to the consumer.

Properties of Walrasian Demand



– For $x \in x(p,w)$, there is a bundle y, ε–close to x, such that $y \succ x$. Then, $x \notin x(p,w)$.

Properties of Walrasian Demand

3) Convexity/Uniqueness:

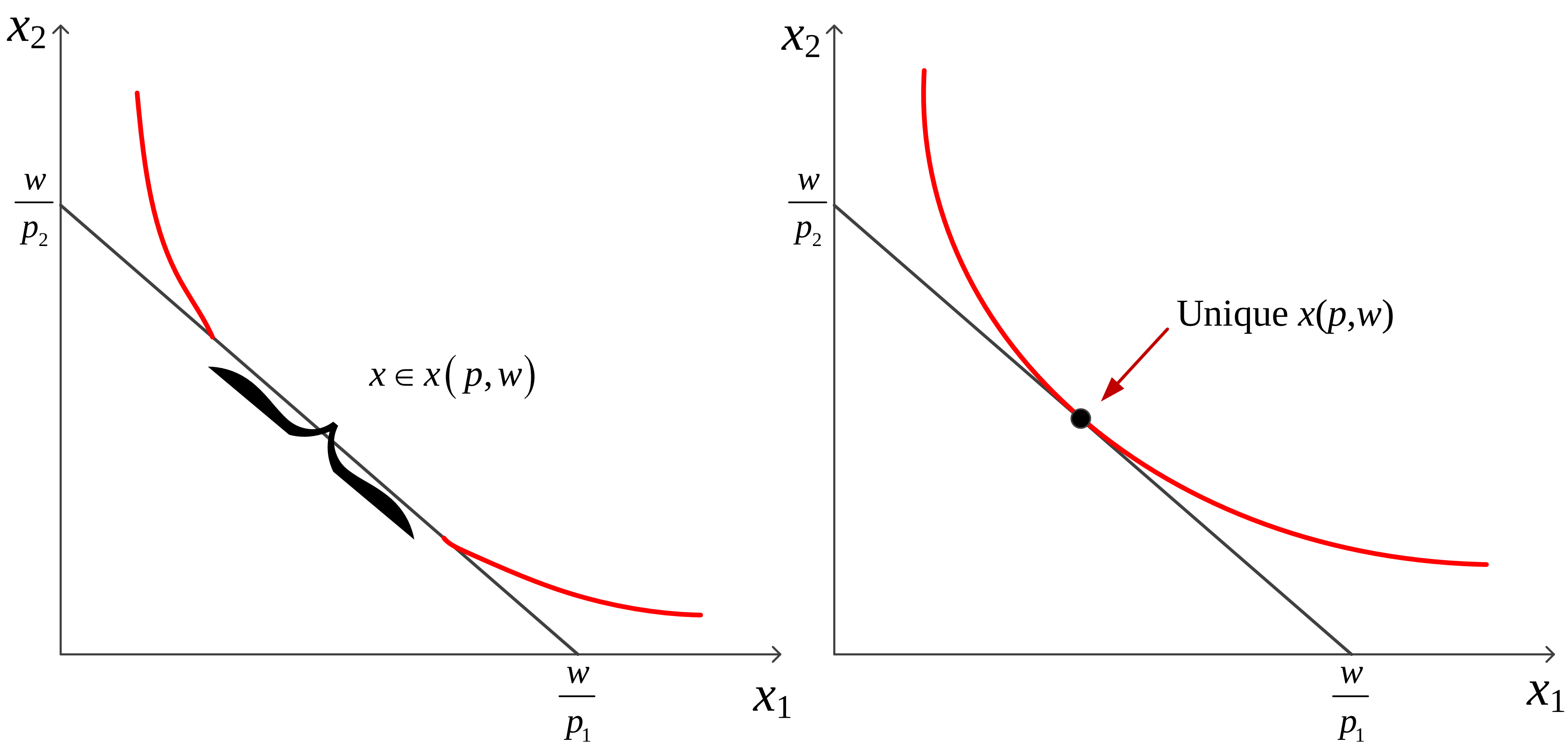
a) If the preferences are convex, then the Walrasian demand correspondence $x(p, w)$ defines a convex set, i.e., a continuum of bundles are utility maximizing.

b) If the preferences are strictly convex, then the Walrasian demand correspondence $x(p, w)$ contains a single element.

Properties of Walrasian Demand

Convex preferences

Strictly convex preferences



UMP: Necessary Condition

$$\max_{x \geq 0} u(x) \quad \text{s.t. } p \cdot x \leq w$$

- We solve it using Kuhn-Tucker conditions over the Lagrangian $L = u(x) + \lambda(w - p \cdot x)$,

$$\frac{\partial L}{\partial x_k} = \frac{\partial u(x^*)}{\partial x_k} - \lambda p_k \leq 0 \text{ for all } k, \; = 0 \text{ if } x_k^* > 0$$

$$\frac{\partial L}{\partial \lambda} = w - p \cdot x^* = 0$$

- That is, in a *interior* optimum, $\frac{\partial u(x^*)}{\partial x_k} = \lambda p_k$ for every good k, which implies

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k} \Leftrightarrow MRS_{\,l,k} = \frac{p_l}{p_k} \Leftrightarrow \frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

UMP: Sufficient Condition

- When are Kuhn-Tucker (necessary) conditions, also sufficient?
  - That is, when can we guarantee that $x(p, w)$ is the $\max$ of the UMP and not the $\min$?

UMP: Sufficient Condition

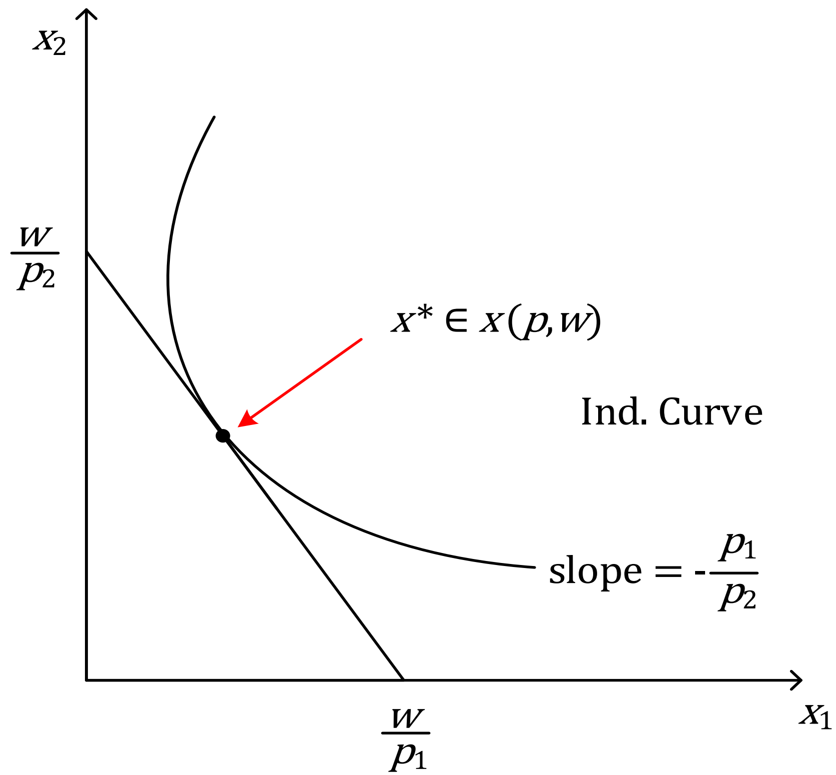
- Kuhn-Tucker conditions are sufficient for a max if:

  1) $u(x)$ is quasiconcave, i.e., convex upper contour set (UCS).

  2) $u(x)$ is monotone.

  3) $\nabla u(x) \neq 0$ for $x \in \mathbb{R}_{+}^{L}$.

  – If $\nabla u(x) = 0$ for some x, then we would be at the "top of the mountain" (i.e., blissing point), which violates both LNS and monotonicity.

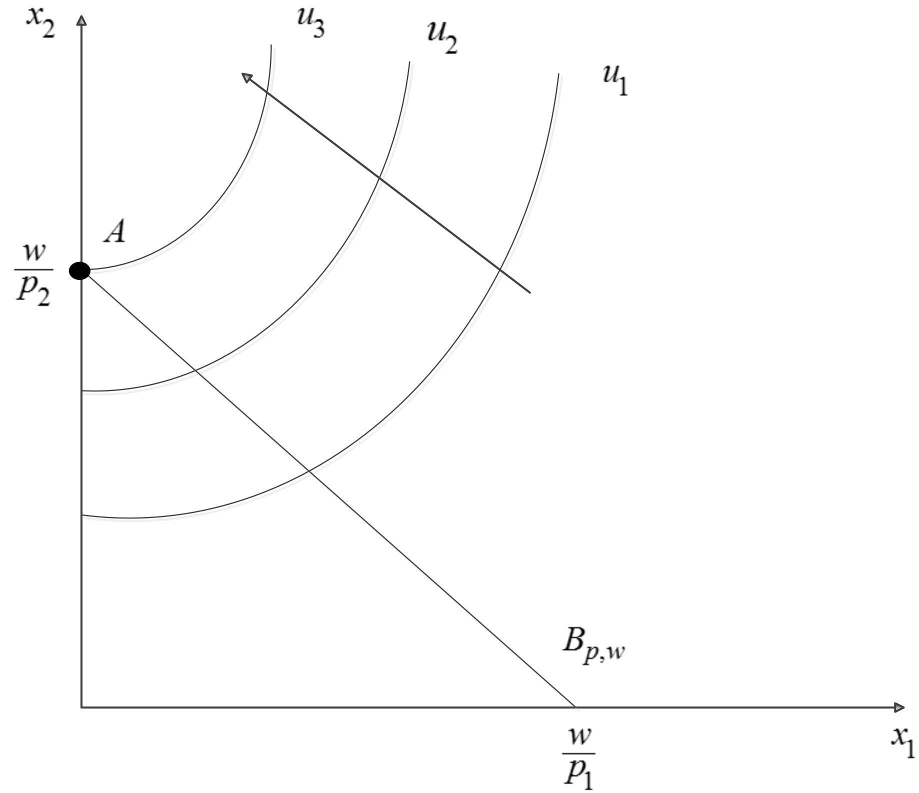


UMP: Violations of Sufficient Condition

1) $u(\cdot)$ ***is non-monotone:***

- The consumer chooses bundle *A* (at a corner) since it yields the highest utility level given his budget constraint.

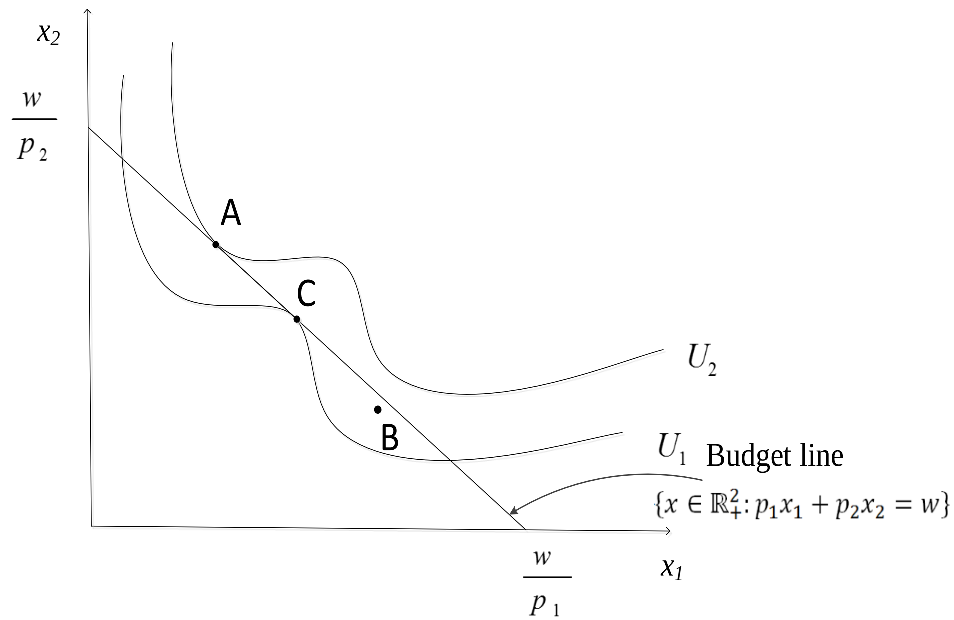
- At point *A*, however, the tangency condition $MRS_{1,2} = \frac{p_1}{p_2}$ does not hold.



UMP: Violations of Sufficient Condition

2) $u(\cdot)$ is not quasiconcave:

- The upper contour sets (UCS) are not convex.
- $MRS_{1,2} = \frac{p_1}{p_2}$ is not a sufficient condition for a max.
- A point of tangency (*C*) gives a lower utility level than a point of non-tangency (*B*).
- True maximum is at point *A*.



UMP: Corner Solution

- Analyzing differential changes in x_l and x_l, that keep individual's utility unchanged, $du = 0$,

$$\frac{du(x)}{dx_l} dx_l + \frac{du(x)}{dx_k} dx_k = 0 \text{ (total diff.)}$$

- Rearranging,

$$\frac{dx_k}{dx_l} = -\frac{\frac{du(x)}{dx_l}}{\frac{du(x)}{dx_k}} = -MRS_{l,k}$$

- ***Corner Solution***: $MRS_{l,k} > \frac{p_l}{p_k}$, or alternatively, $\frac{\frac{du(x^*)}{dx_l}}{p_l} > \frac{\frac{du(x^*)}{dx_k}}{p_k}$, i.e., the consumer prefers to consume more of good l.

UMP: Corner Solution

- In the FOCs, this implies:
  - *a)* $\frac{\partial u(x^*)}{\partial x_k} \leq \lambda p_k$ for the goods whose consumption is zero, $x_k^* = 0$, and
  - *b)* $\frac{\partial u(x^*)}{\partial x_l} = \lambda p_l$ for the good whose consumption is positive, $x_l^* > 0$.
- *Intuition*: the marginal utility per dollar spent on good l is still larger than that on good k.

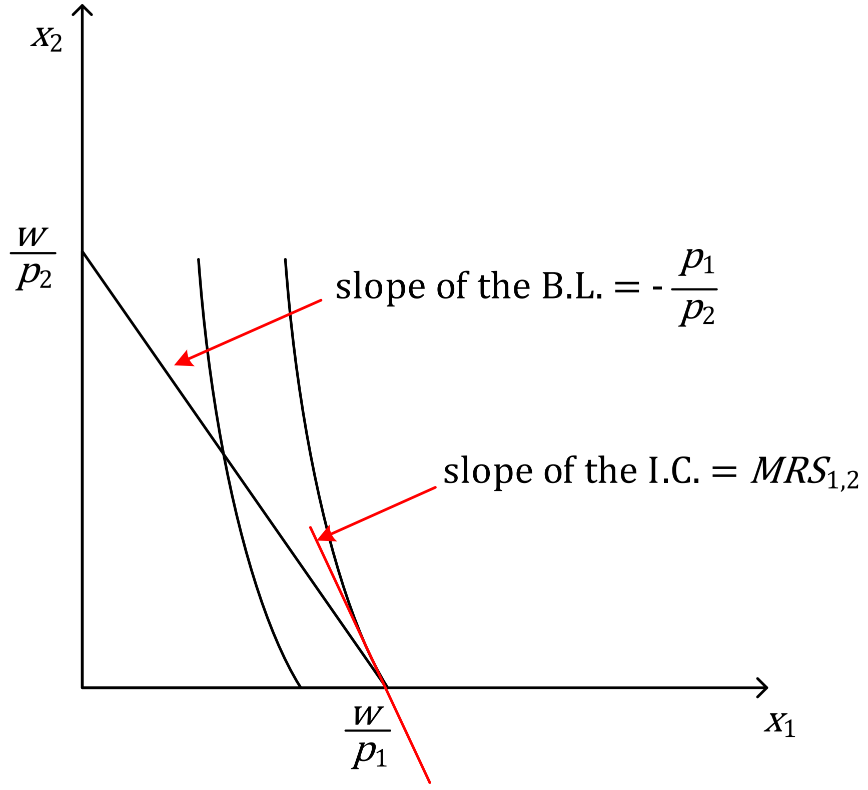
$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} = \lambda \geq \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

UMP: Corner Solution

- Consumer seeks to consume good 1 alone.
- At the corner solution, the indifference curve is steeper than the budget line, i.e.,

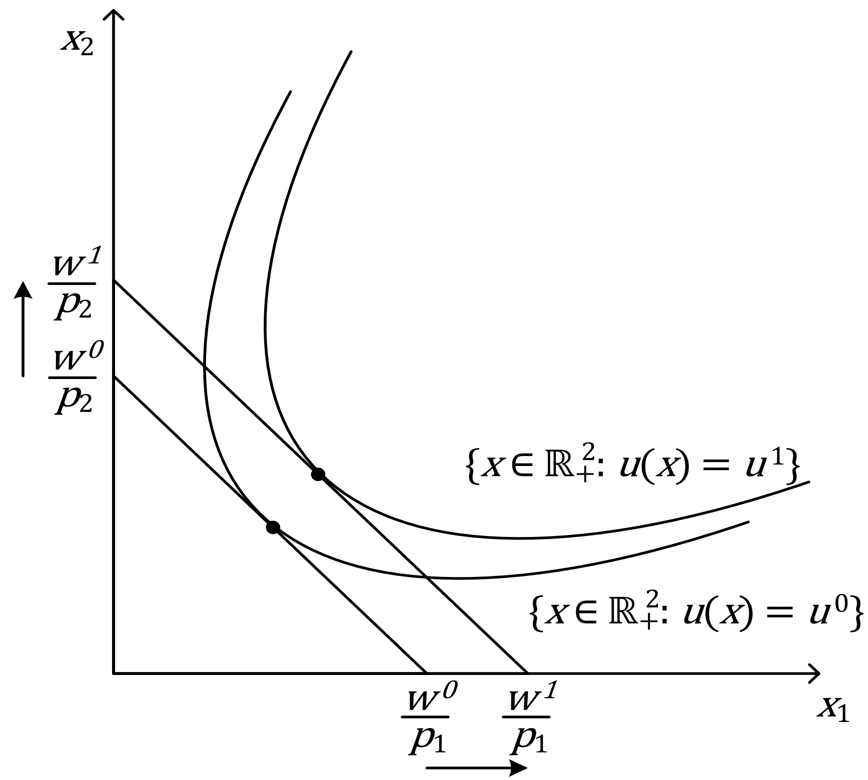
$$MRS_{1,2} > \frac{p_1}{p_2} \text{ or } \frac{MU_1}{p_1} > \frac{MU_2}{p_2}$$

- Intuitively, the consumer would like to consume more of good 1, even after spending his entire wealth on good 1 alone.



UMP: Lagrange Multiplier

- λ is referred to as the "marginal values of relaxing the constraint" in the UMP (a.k.a. "***shadow price of wealth***").
- If we provide more wealth to the consumer, he is capable of reaching a higher indifference curve and, as a consequence, obtaining a higher utility level.
  - We want to measure the change in utility resulting from a marginal increase in wealth.



UMP: Lagrange Multiplier

- Let us take $u(x(p, w))$, and analyze the change in utility from change in wealth. Using the chain rule yields,

$$\nabla u(x(p, w)) \cdot D_w x(p, w)$$

- Substituting $\nabla u(x(p, w)) = \lambda p$ (in interior solutions),

$$\lambda p \cdot D_w x(p, w)$$

UMP: Lagrange Multiplier

- From Walras' Law, $p \cdot x(p,w) = w$, the change in expenditure from an increase in wealth is given by
$$D_w[p \cdot x(p,w)] = p \cdot D_w x(p,w) = 1$$

- Hence,
$$\nabla u(x(p,w)) \cdot D_w x(p,w) = \lambda \underbrace{p \cdot D_w x(p,w)}_{1} = \lambda$$

- *Intuition*: If $\lambda = 5$, then a \$1 increase in wealth implies an increase in 5 units of utility.

Walrasian Demand: Wealth Effects

- Normal vs. Inferior goods

$$\frac{\partial x(p,w)}{\partial w} \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{normal} \\ \text{inferior} \end{Bmatrix}$$

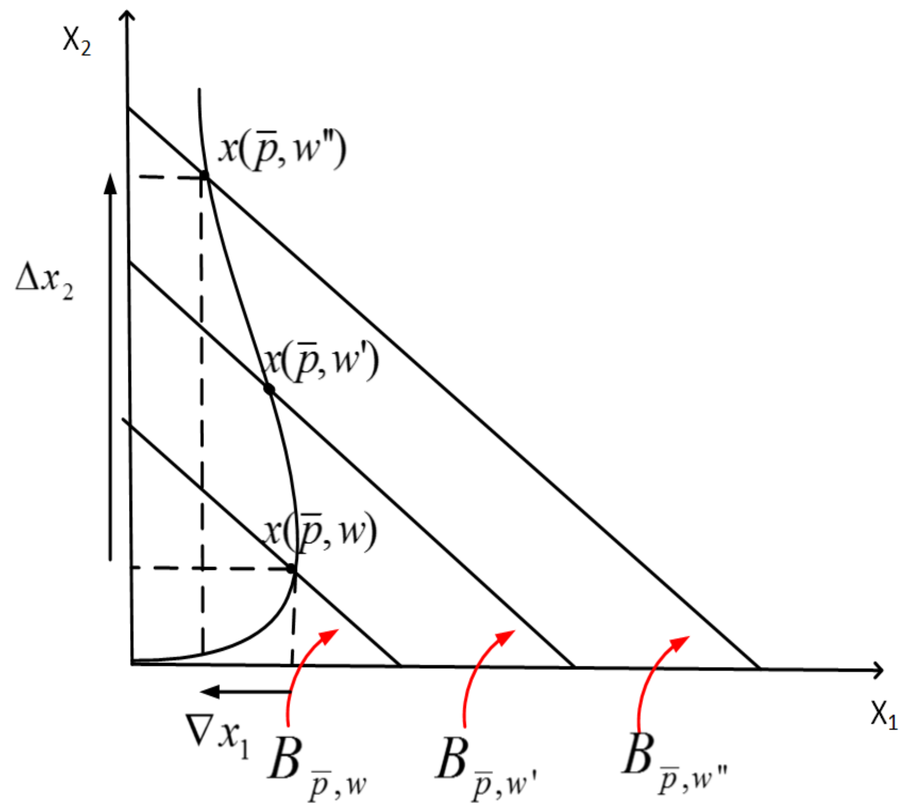
- Examples of inferior goods:
  – Two-buck chuck (a really cheap wine)
  – Walmart during the economic crisis

Walrasian Demand: Wealth Effects

- An increase in the wealth level produces an outward shift in the budget line.

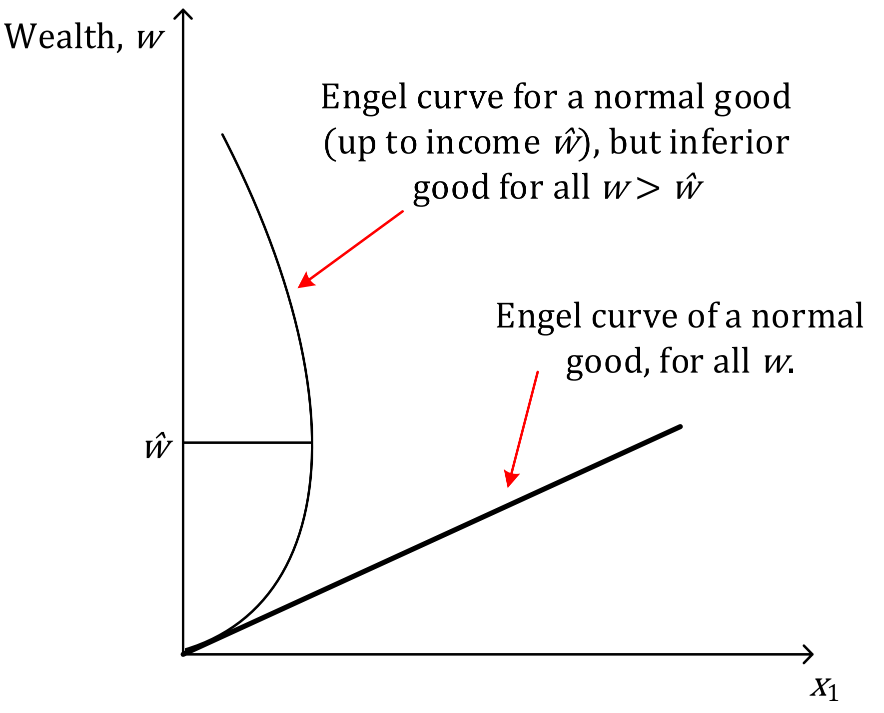
- x_2 is normal as $\frac{\partial x_2(p,w)}{\partial w} > 0$, while x_1 is inferior as $\frac{\partial x(p,w)}{\partial w} < 0$.

- ***Wealth expansion path:***
  - connects the optimal consumption bundle for different levels of wealth
  - indicates how the consumption of a good changes as a consequence of changes in the wealth level



Walrasian Demand: Wealth Effects

- ***Engel curve*** depicts the consumption of a particular good in the horizontal axis and wealth on the vertical axis.
- The slope of the Engel curve is:
  - positive if the good is normal
  - negative if the good is inferior
- Engel curve can be positively slopped for low wealth levels and become negatively slopped afterwards.



Walrasian Demand: Price Effects

- Own price effect:

$$\frac{\partial x_k(p,w)}{\partial p_k} \begin{Bmatrix} < \\ > \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{Usual} \\ \text{Giffen} \end{Bmatrix}$$

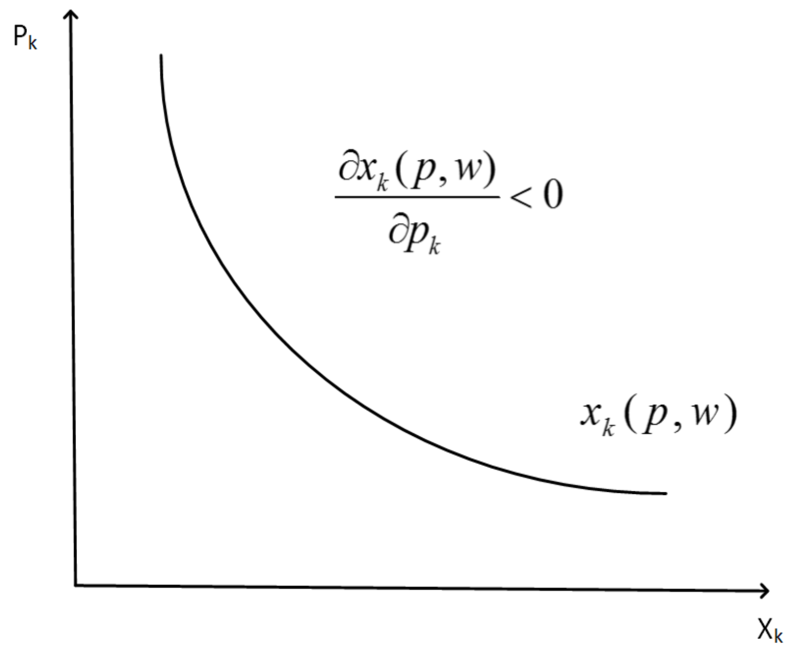
- Cross-price effect:

$$\frac{\partial x_k(p,w)}{\partial p_l} \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{Substitutes} \\ \text{Complements} \end{Bmatrix}$$

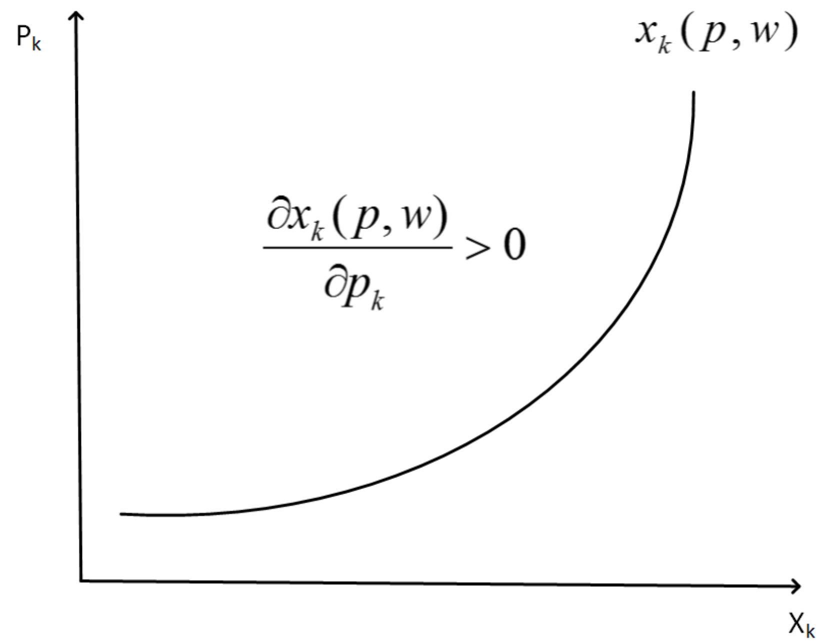
  – *Examples of Substitutes*: two brands of mineral water, such as Aquafina vs. Poland Springs.
  – *Examples of Complements*: cars and gasoline.

Walrasian Demand: Price Effects

- Own price effect



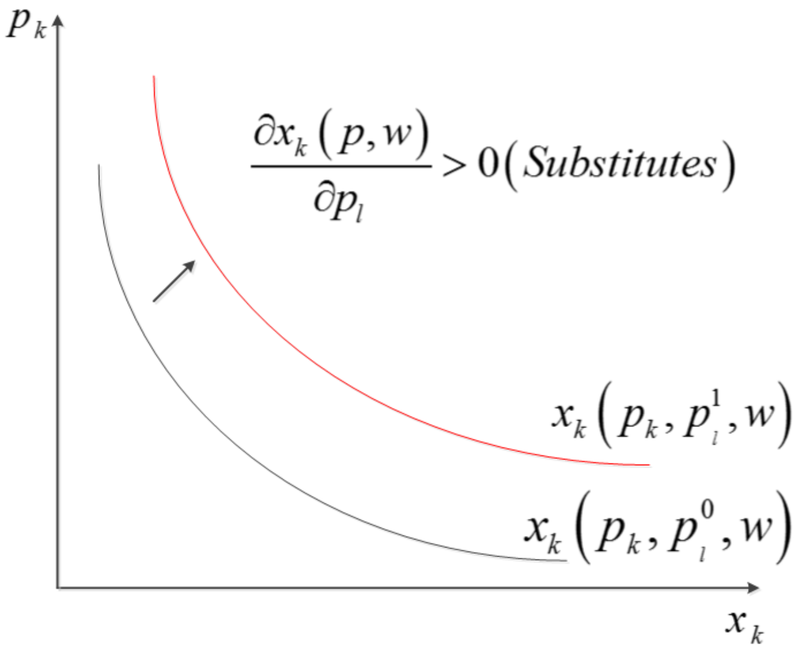
Usual good



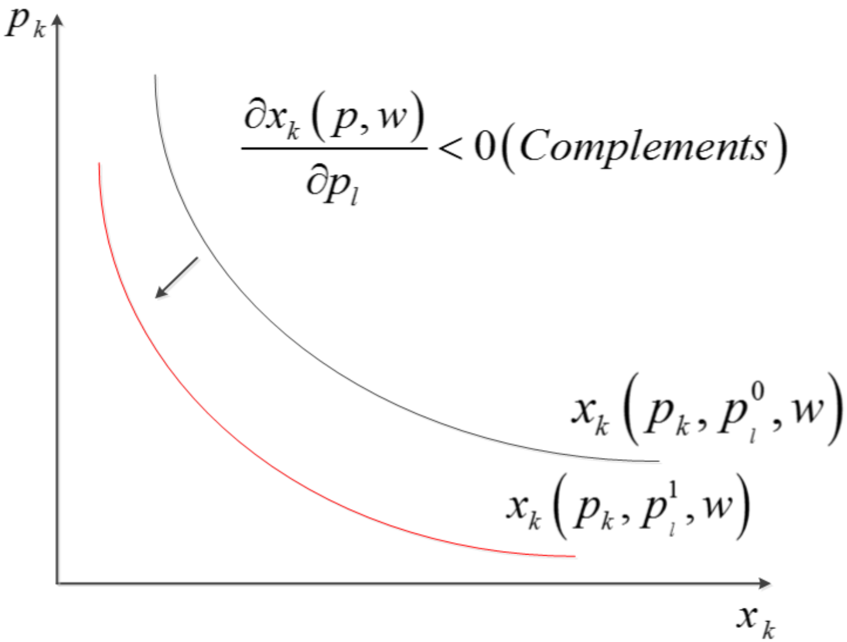
Giffen good

Walrasian Demand: Price Effects

- Cross-price effect



Substitutes



Complements

Indirect Utility Function

- The Walrasian demand function, $x(p, w)$, is the solution to the UMP (i.e., argmax).

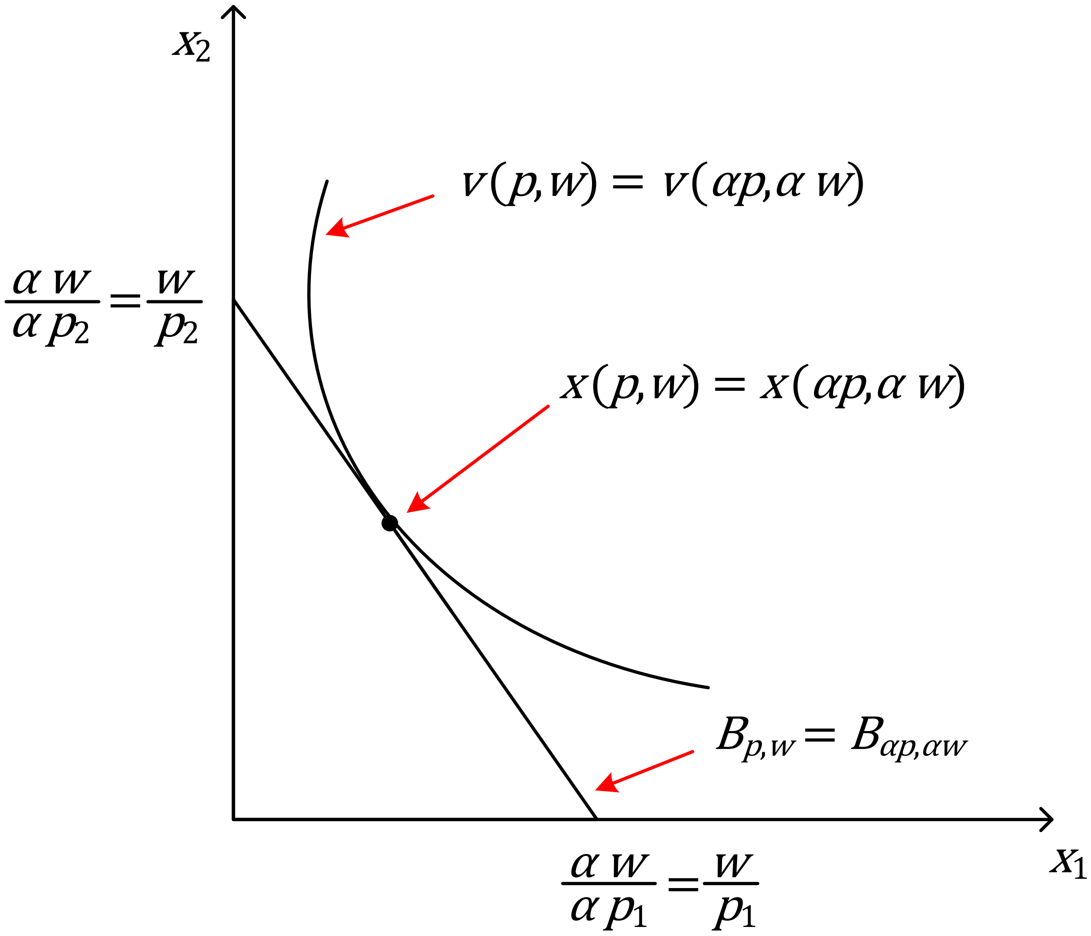
- What would be the utility function evaluated at the solution of the UMP, i.e., $x(p, w)$?
  - This is the ***indirect utility function*** (i.e., the highest utility level), $v(p, w) \in \mathbb{R}$, associated with the UMP.
  - It is the "value function" of this optimization problem.

Properties of Indirect Utility Function

- If the utility function is continuous and preferences satisfy LNS over the consumption set $X = \mathbb{R}^L_+$, then the indirect utility function $v(p, w)$ satisfies:

  1) ***Homogenous of degree zero:*** Increasing p and w by a common factor $\alpha > 0$ does not modify the consumer's optimal consumption bundle, $x(p, w)$, nor his maximal utility level, measured by $v(p, w)$.

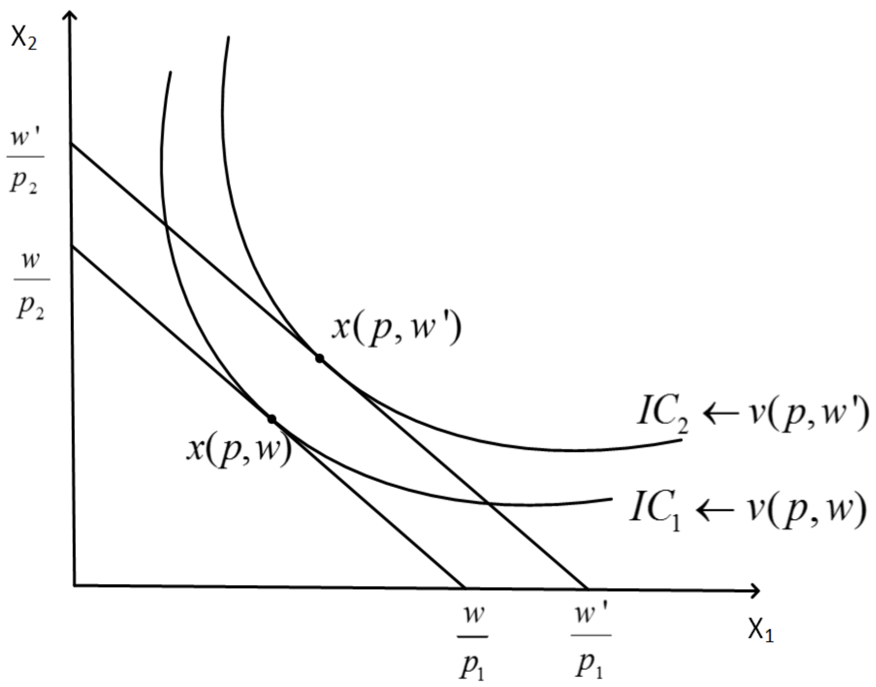
Properties of Indirect Utility Function



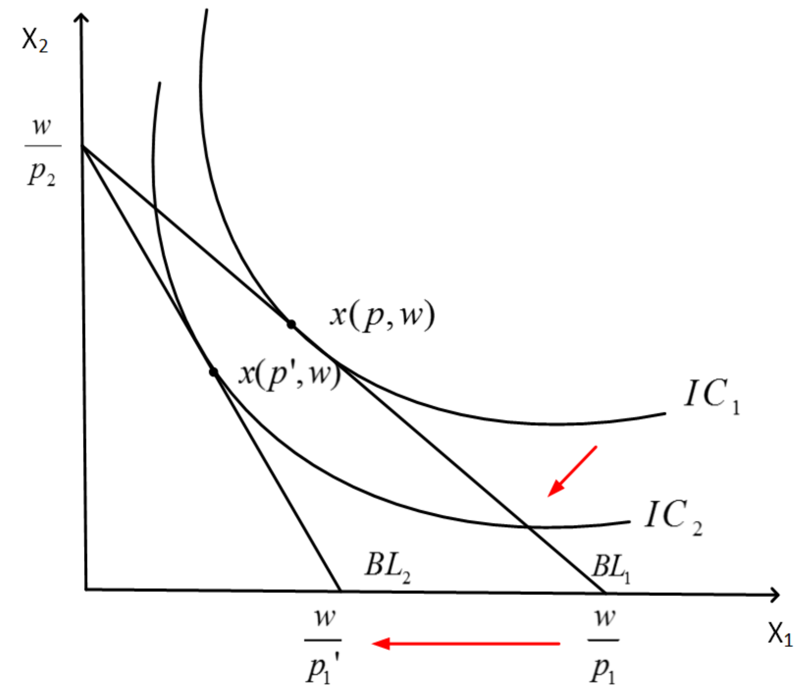
Properties of Indirect Utility Function

2) Strictly increasing in w:

$v(p, w') > v(p, w)$ for $w' > w$.



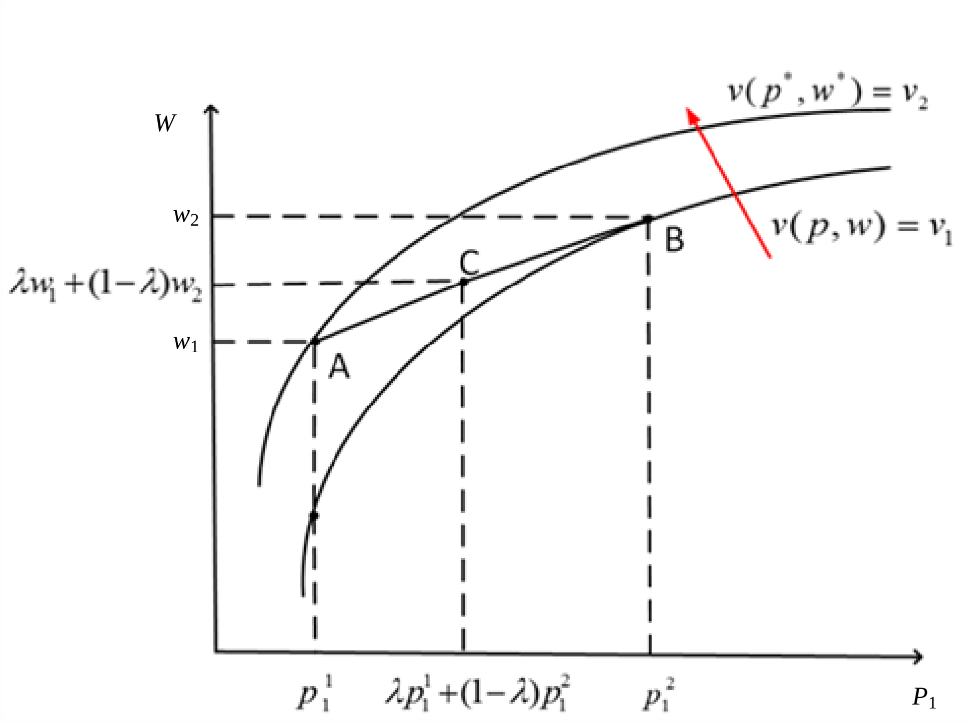
3) non-increasing (i.e., weakly decreasing) in p_k



Properties of Indirect Utility Function

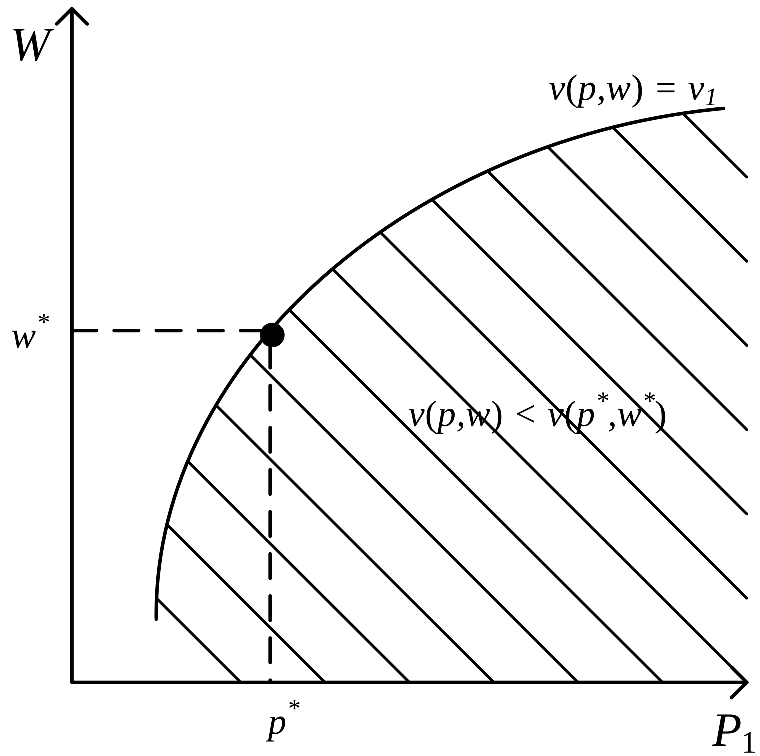
4) Quasiconvex: The set $\{(p, w): v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

- ***Interpretation I***: If $(p^1, w^1) \gtrsim^* (p^2, w^2)$, then $(p^1, w^1) \gtrsim^* (\lambda p^1 + (1-\lambda)p^2,\ \lambda w^1 + (1-\lambda)w^2)$; i.e., if $A \gtrsim^* B$, then $A \gtrsim^* C$.



Properties of Indirect Utility Function

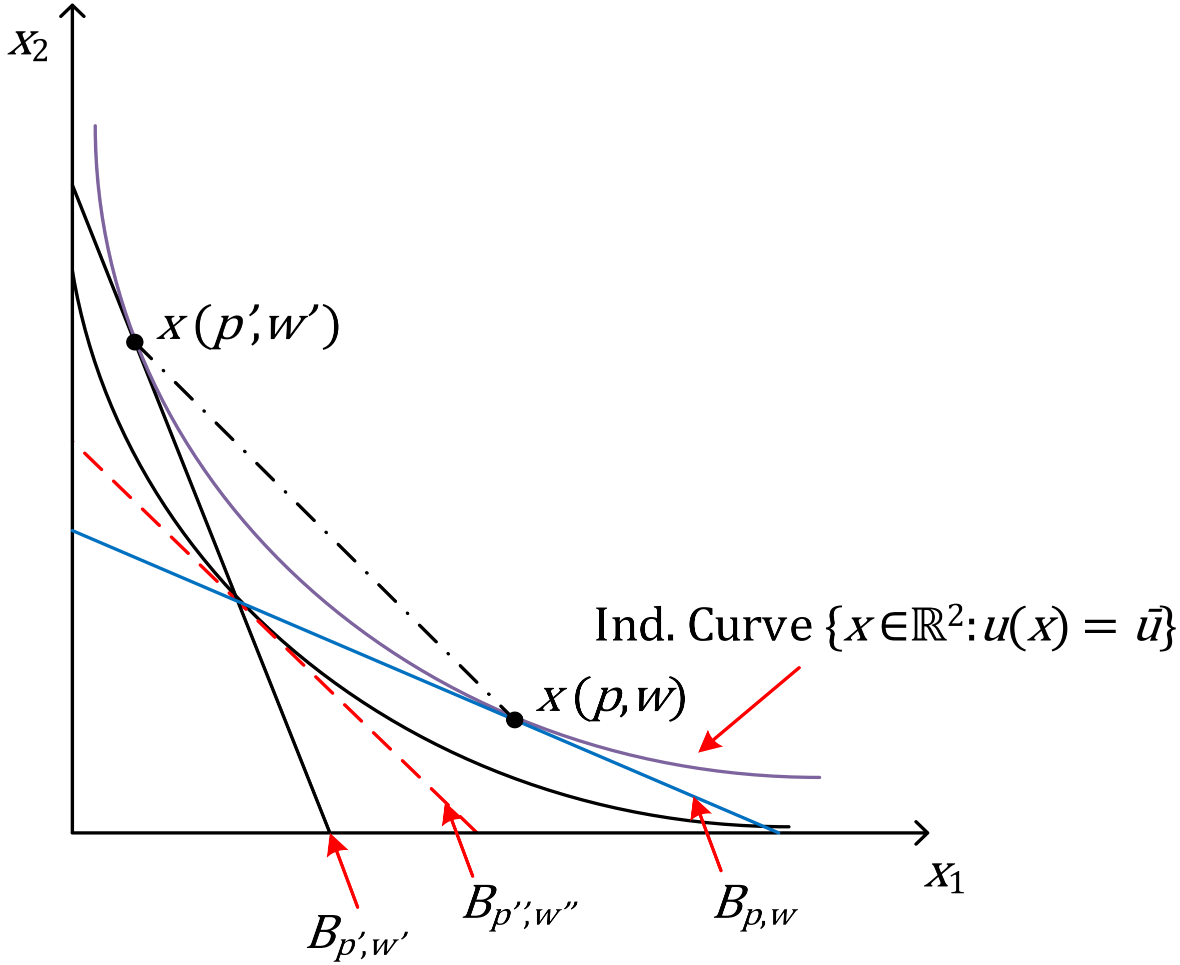
- ***Interpretation II***: $v(p, w)$ is quasiconvex if the set of (p, w) pairs for which $v(p, w) < v(p^*, w^*)$ is convex.



Properties of Indirect Utility Function

- ***Interpretation III***: Using x_1 and x_2 in the axis, perform following steps:
    1) When $B_{p,w}$, then $x(p,w)$
    2) When $B_{p',w'}$, then $x(p',w')$
    3) Both $x(p,w)$ and $x(p',w')$ induce an indirect utility of $v(p,w) = v(p',w') = \bar{u}$
    4) Construct a linear combination of prices and wealth:
    $$\left.\begin{aligned} p'' &= \alpha p + (1-\alpha)p' \\ w'' &= \alpha w + (1-\alpha)w' \end{aligned}\right\} \quad B_{p'',w''}$$
    5) Any solution to the UMP given $B_{p'',w''}$ must lie on a lower indifference curve (i.e., lower utility)
    $$v(p'',w'') \leq \bar{u}$$

Properties of Indirect Utility Function



WARP and Walrasian Demand

- Relation between Walrasian demand $x(p, w)$ and WARP
  - How does the WARP restrict the set of optimal consumption bundles that the individual decision-maker can select when solving the UMP?

WARP and Walrasian Demand

- Take two different consumption bundles $x(p,w)$ and $x(p',w')$, both being affordable (p,w), i.e.,

$$p \cdot x(p,w) \leq w \text{ and } p \cdot x(p',w') \leq w$$

- When prices and wealth are (p,w), the consumer chooses $x(p,w)$ despite $x(p',w')$ is also affordable.
- Then he "reveals" a preference for $x(p,w)$ over $x(p',w')$ when both are affordable.
- Hence, we should expect him to choose $x(p,w)$ over $x(p',w')$ when both are affordable. (Consistency)
- Therefore, bundle $x(p,w)$ must not be affordable at (p',w') because the consumer chooses $x(p',w')$. That is, $p' \cdot x(p,w) > w'$.

WARP and Walrasian Demand

- In summary, Walrasian demand satisfies WARP, if, for two different consumption bundles, $x(p,w) \neq x(p',w')$,

$$p \cdot x(p',w') \leq w \quad \Rightarrow \quad p' \cdot x(p,w) > w'$$

- *Intuition*: if $x(p',w')$ is affordable under budget set $B_{p,w}$, then $x(p,w)$ cannot be affordable under $B_{p',w'}$.

Checking for WARP

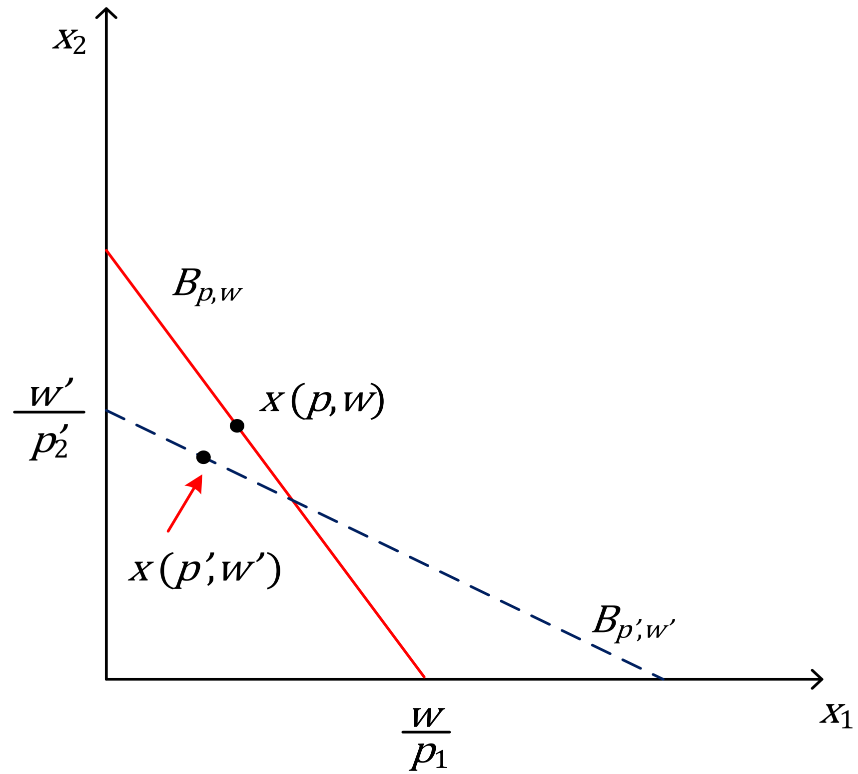
- A systematic procedure to check if Walrasian demand satisfies WARP:
  - ***Step 1:*** Check if bundles $x(p, w)$ and $x(p', w')$ are both affordable under $B_{p,w}$.
    - That is, graphically $x(p, w)$ and $x(p', w')$ have to lie on or below budget line $B_{p,w}$.
    - If step 1 is satisfied, then move to step 2.
    - Otherwise, the premise of WARP does not hold, which does not allow us to continue checking if WARP is violated or not. In this case, we can only say that "*WARP is not violated*".

Checking for WARP

- ***Step 2:*** Check if bundles $x(p, w)$ is affordable under $B_{p',w'}$.
  - That is, graphically $x(p, w)$ must lie on or below budge line $B_{p',w'}$.
  - If step 2 is satisfied, then this Walrasian demand violates WARP.
  - Otherwise, the Walrasian demand satisfies WARP.

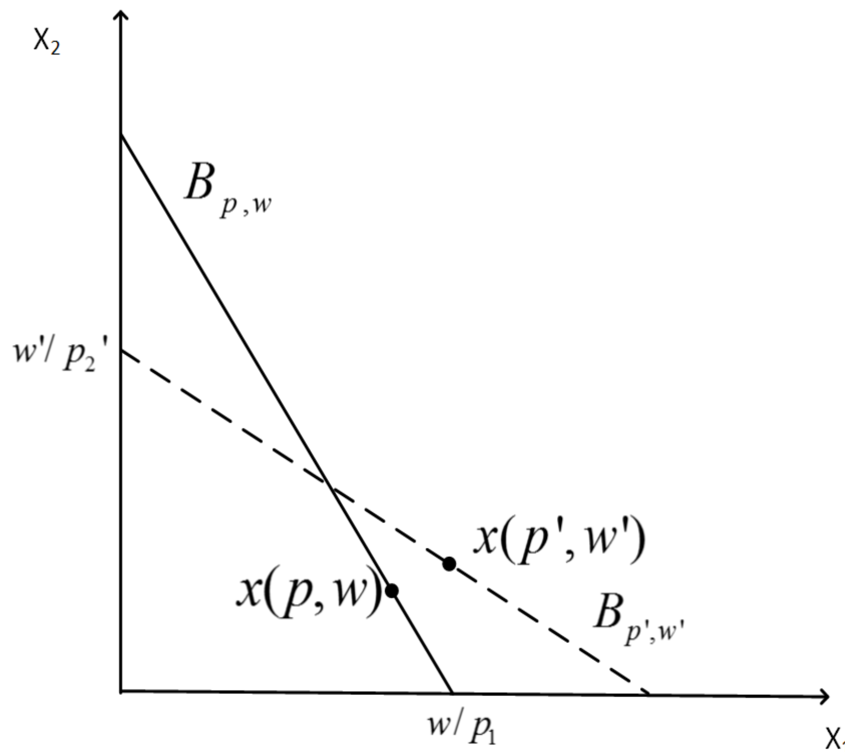
Checking for WARP: Example 1

- First, $x(p, w)$ and $x(p', w')$ are both affordable under $B_{p,w}$.
- Second, $x(p, w)$ is not affordable under $B_{p',w'}$.
- Hence, WARP is *satisfied*!



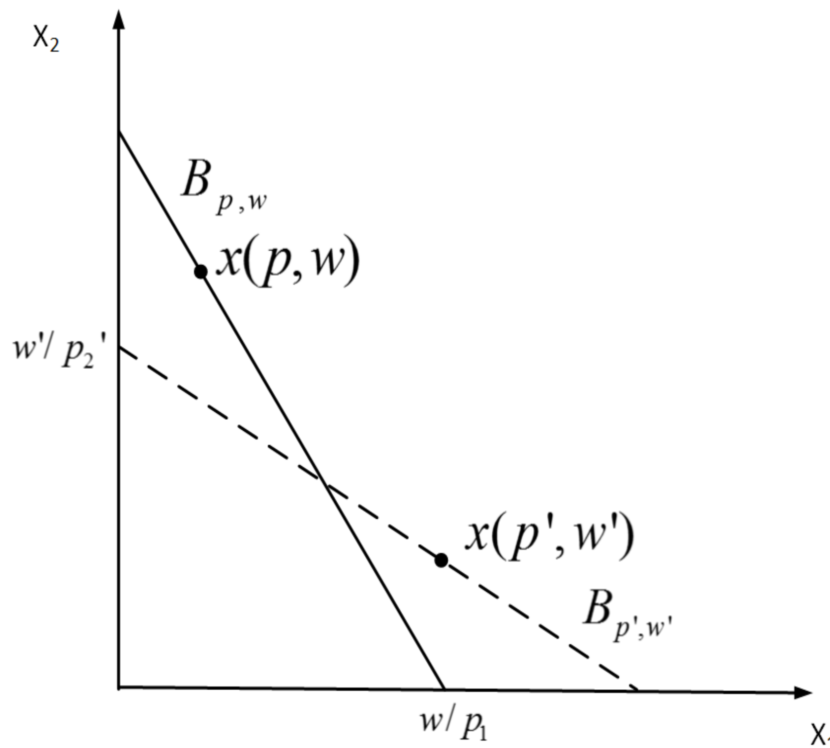
Checking for WARP: Example 2

- The demand $x(p', w')$ under final prices and wealth is not affordable under initial prices and wealth, i.e., $p \cdot x(p', w') > w$.
  - The premise of WARP does not hold.
  - Violation of Step 1!
  - WARP is *not violated*.



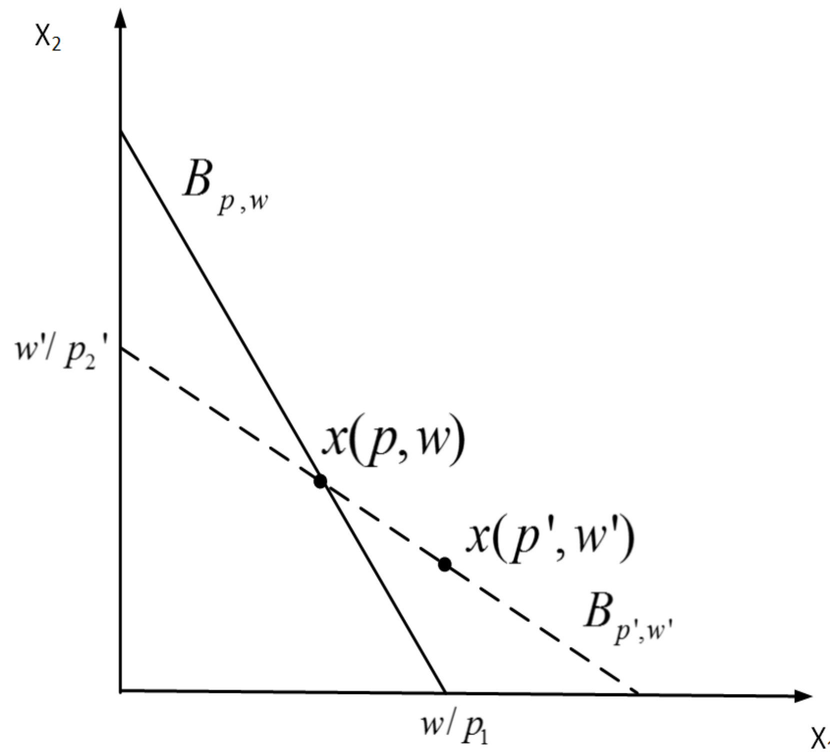
Checking for WARP: Example 3

- The demand $x(p', w')$ under final prices and wealth is not affordable under initial prices and wealth, i.e., $p \cdot x(p', w') > w$.
  - The premise of WARP does not hold.
  - Violation of Step 1!
  - WARP is *not violated*.



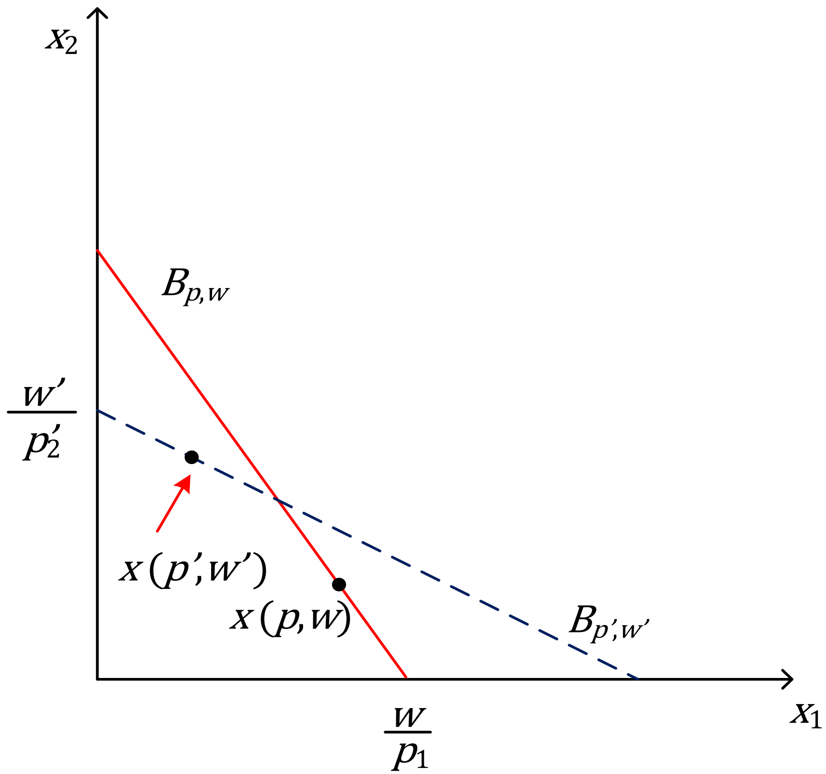
Checking for WARP: Example 4

- The demand $x(p', w')$ under final prices and wealth is not affordable under initial prices and wealth, i.e., $p \cdot x(p', w') > w$.
  - The premise of WARP does not hold.
  - Violation of Step 1!
  - WARP is *not violated*.



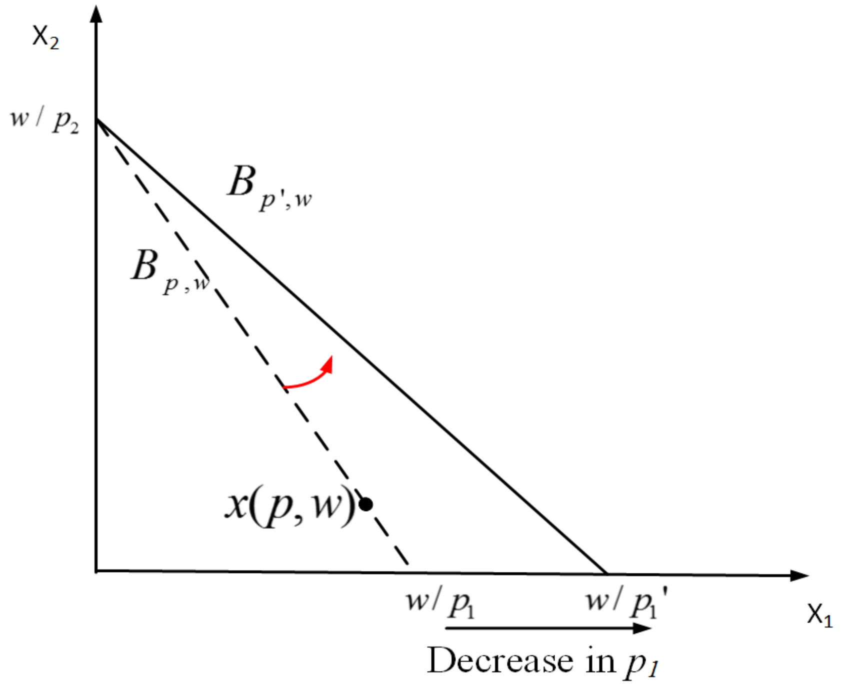
Checking for WARP: Example 5

- First, $x(p, w)$ and $x(p', w')$ are both affordable under $B_{p,w}$.
- Second, $x(p, w)$ is affordable under $B_{p',w'}$, i.e., $p' \cdot x(p, w) < w'$
- Hence, WARP is *NOT satisfied*!



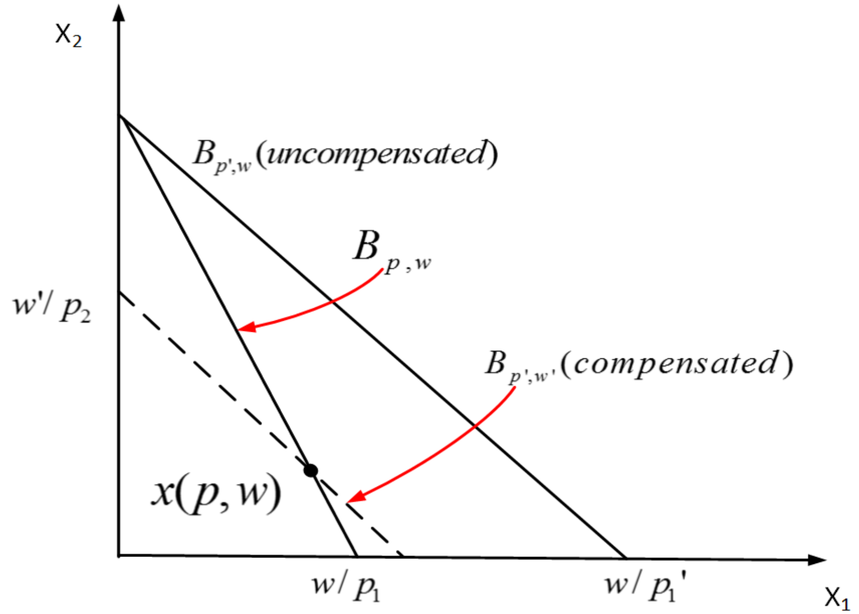
Implications of WARP

- How do price changes affect the WARP predictions?
- Assume a reduction in p_1
  - the consumer's budget lines rotates (***uncompensated price change***)



Implications of WARP

- Adjust the consumer's wealth level so that he can consume his initial demand $x(p, w)$ at the new prices.
  - shift the final budget line inwards until the point at which we reach the initial consumption bundle $x(p, w)$ (***compensated price change***)



Implications of WARP

- What is the wealth adjustment?

$$w = p \cdot x(p, w) \text{ under } (p, w)$$
$$w' = p' \cdot x(p, w) \text{ under } (p', w')$$

- Then,

$$\Delta w = \Delta p \cdot x(p, w)$$

where $\Delta w = w' - w$ and $\Delta p = p' - p$.

- This is the ***Slutsky wealth compensation:***
  - the increase (decrease) in wealth, measured by Δw, that we must provide to the consumer so that he can afford the same consumption bundle as before the price increase (decrease), $x(p, w)$.

Implications of WARP: Law of Demand

- Suppose that the Walrasian demand $x(p,w)$ satisfies $\text{homog}(0)$ and Walras' Law. Then, $x(p,w)$ satisfies WARP iff:

$$\Delta p \cdot \Delta x \leq 0$$

  where
  - $\Delta p = p' - p$ and $\Delta x = x(p', w') - x(p, w)$
  - w' is the wealth level that allows the consumer to buy the initial demand at the new prices, $w' = p' \cdot x(p, w)$

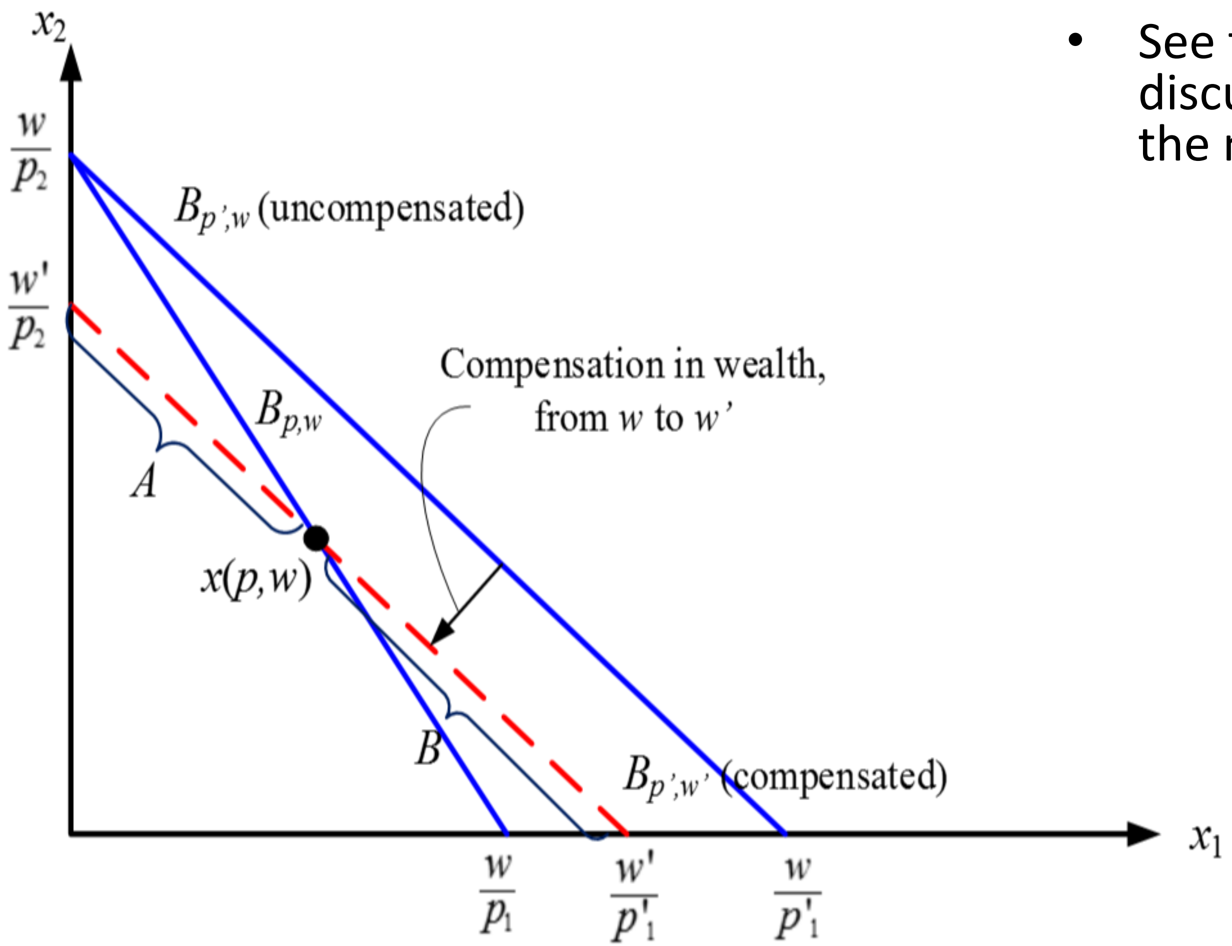
- This is the ***Law of Demand***: quantity demanded and price move in different directions.

Implications of WARP: Law of Demand

- Does WARP restrict behavior when we apply Slutsky wealth compensations?
  - Yes!

- What if we were not applying the Slutsky wealth compensation, would WARP impose any restriction on allowable locations for $x(p', w')$?
  - No!

Implications of WARP: Law of Demand



- See the discussions on the next slide

Implications of WARP: Law of Demand

- Can $x(p', w')$ lie on segment *A*?
  1) $x(p, w)$ and $x(p', w')$ are both affordable under $B_{p,w}$.
  2) $x(p, w)$ is affordable under $B_{p',w'}$.
     - ⇒ WARP is violated if $x(p', w')$ lies on segment *A*

- Can $x(p', w')$ lie on segment *B*?
  1) $x(p, w)$ is affordable under $B_{p,w}$, but $x(p', w')$ is not.
     - ⇒ The premise of WARP does not hold
     - ⇒ WARP is not violated if $x(p', w')$ lies on segment *B*

Implications of WARP: Law of Demand

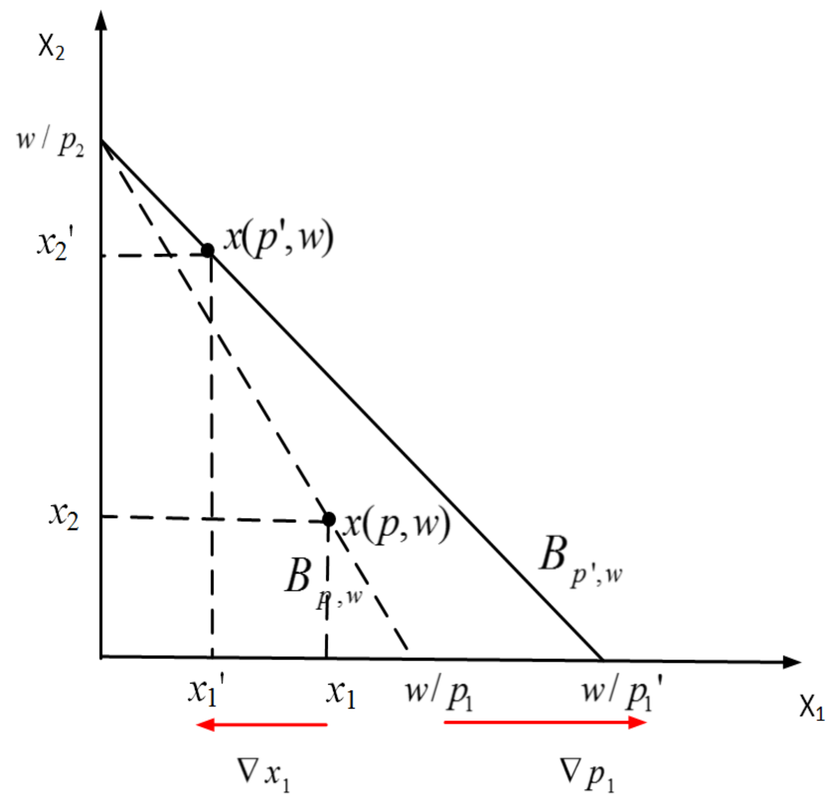
- What did we learn from this figure?
  1) We started from ∇p_1, and compensated the wealth of this individual (reducing it from w to w') so that he could afford his initial bundle $x(p, w\ )$ under the new prices.
     - From this wealth compensation, we obtained budget line $B_{p',w'}$.
  2) From WARP, we know that $x(p', w')$ must contain more of good 1.
     - That is, graphically, segment *B* lies to the right-hand side of $x(p, w\ )$.
  3) Then, a price reduction, ∇p_1, when followed by an appropriate wealth compensation, leads to an increase in the quantity demanded of this good, Δx_1.
     - This is the ***compensated law of demand*** (CLD).

Implications of WARP: Law of Demand

- *Practice problem*:
  - Can you repeat this analysis but for an increase in the price of good 1?
    - First, pivot budget line $B_{p,w}$ inwards to obtain $B_{p',w}$.
    - Then, increase the wealth level (wealth compensation) to obtain $B_{p',w'}$.
    - Identify the two segments in budget line $B_{p',w'}$, one to the right-hand side of $x(p, w\ )$ and other to the left.
    - In which segment of budget line $B_{p',w'}$ can the Walrasian demand $x(p', w')$ lie?

Implications of WARP: Law of Demand

- Is WARP satisfied under the ***uncompensated law of demand*** (ULD)?
  - $x(p, w)$ is affordable under budget line $B_{p,w}$, but $x(p', w)$ is not.
    - Hence, the premise of WARP is not satisfied. As a result, WARP is not violated.
  - But, is this result implying something about whether ULD must hold?
    - No!
    - Although WARP is not violated, ULD is: a decrease in the price of good 1 yields a decrease in the quantity demanded.



Implications of WARP: Law of Demand

- Distinction between the uncompensated and the compensated law of demand:
  - quantity demanded and price can move in the same direction, when wealth is left uncompensated, i.e.,

$$\Delta p_1 \cdot \Delta x_1 > 0$$

$$\text{as } \Delta p_1 \Rightarrow \Delta x_1$$

- Hence, WARP is not sufficient to yield law of demand for price changes that are uncompensated, i.e.,

$$\text{WARP} \not\Leftrightarrow \text{ULD, but}$$

$$\text{WARP} \Leftrightarrow \text{CLD}$$

Implications of WARP: Slutsky Matrix

- Let us focus now on the case in which $x(p, w)$ is differentiable.
- First, note that the law of demand is $dp \cdot dx \leq 0$ (equivalent to $\Delta p \cdot \Delta x \leq 0$).
- Totally differentiating $x(p, w)$,

$$dx = D_p x(p, w) dp + D_w x(p, w) dw$$

- And since the consumer's wealth is compensated, $dw = x(p, w) \cdot dp$ (this is the differential analog of $\Delta w = \Delta p \cdot x(p, w)$).
  - Recall that $\Delta w = \Delta p \cdot x(p, w)$ was obtained from the Slutsky wealth compensation.

Implications of WARP: Slutsky Matrix

- Substituting,

$$dx = D_p x(p,w)dp + D_w x(p,w) \underbrace{[x(p,w) \cdot dp]}_{dw}$$

or equivalently,

$$dx = \left[D_p x(p,w) + D_w x(p,w) x(p,w)^T\right] dp$$

Implications of WARP: Slutsky Matrix

- Hence the law of demand, $dp \cdot dx \leq 0$, can be expressed as

$$dp \cdot \underbrace{\left[D_p x(p,w) + D_w x(p,w) x(p,w)^T\right] dp}_{dx} \leq 0$$

where the term in brackets is the ***Slutsky (or substitution) matrix***

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix}$$

where each element in the matrix is

$$s_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

Implications of WARP: Slutsky Matrix

- ***Proposition:*** If $x(p, w)$ is differentiable, satisfies Walras' law, $\text{homog}(0)$, and WARP, then $S(p, w)$ is negative semi-definite,

$$v \cdot S(p, w)v \leq 0 \text{ for any } v \in \mathbb{R}^L$$

- Implications:
  - $s_{lk}(p, w)$: substitution effect of good l with respect to its own price is non-positive (own-price effect)
  - Negative semi-definiteness does not imply that $S(p, w)$ is symmetric (except when $L = 2$).
    - Usual confusion: "then $S(p, w)$ is not symmetric", NO!

Implications of WARP: Slutsky Matrix

- ***Proposition:*** If preferences satisfy LNS and strict convexity, and they are represented with a continuous utility function, then the Walrasian demand $x(p, w)$ generates a Slutsky matrix, $S(p, w)$, which is symmetric.

- The above assumptions are really common.
  - Hence, the Slutsky matrix will then be symmetric.

- However, the above assumptions are not satisfied in the case of preferences over perfect substitutes (i.e., preferences are convex, but not strictly convex).

Implications of WARP: Slutsky Matrix

- Non-positive substitution effect, $s_{ll} \leq 0$:

$$\underbrace{s_{ll}(p,w)}_{\text{substitution effect } (-)} = \underbrace{\frac{\partial x_l(p,w)}{\partial p_l}}_{\substack{\text{Total effect:} \\ (-) \text{ usual good} \\ (+) \text{ Giffen good}}} + \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_l(p,w)}_{\substack{\text{Income effect:} \\ (+) \text{ normal good} \\ (-) \text{ inferior good}}}$$

- Substitution Effect = Total Effect + Income Effect

  $\Rightarrow$ Total Effect = Substitution Effect - Income Effect

Implications of WARP: Slutsky Equation

$$\underbrace{s_{ll}(p,w)}_{\text{substitution effect } (-)} = \underbrace{\frac{\partial x_l(p,w)}{\partial p_l}}_{\text{Total effect}} + \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_l(p,w)}_{\text{Income effect}}$$

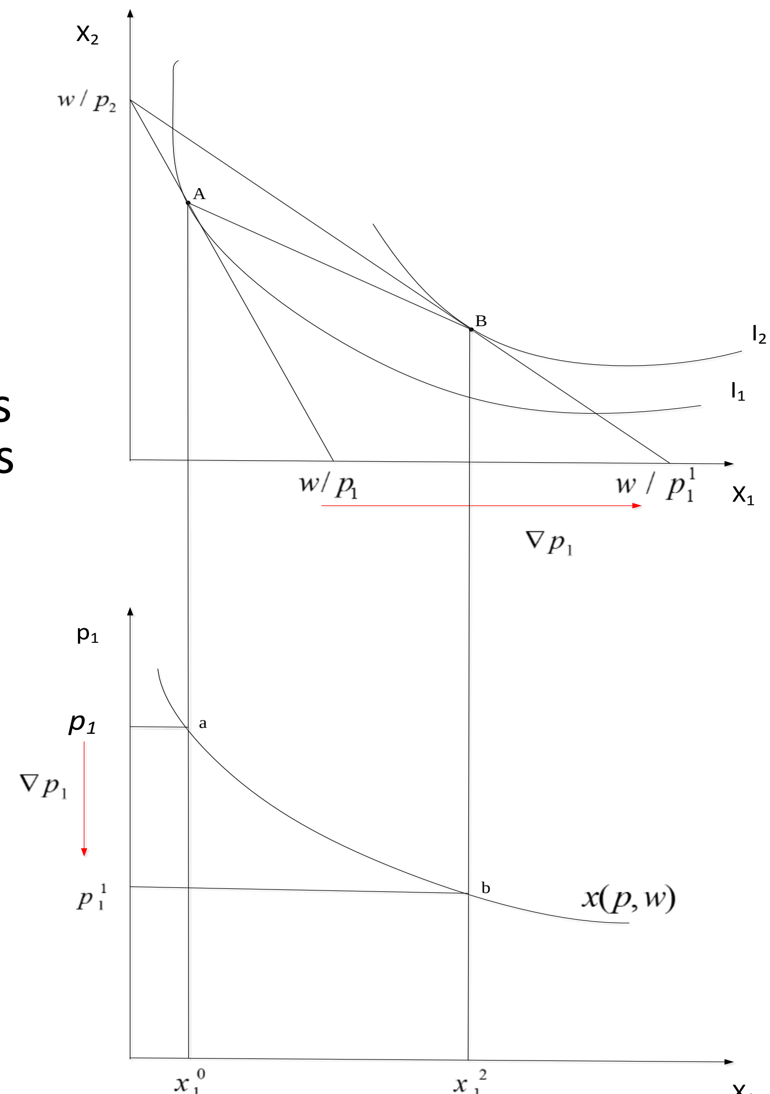
- ***Total Effect:*** measures how the quantity demanded is affected by a change in the price of good l, when we leave the wealth uncompensated.
- ***Income Effect:*** measures the change in the quantity demanded as a result of the wealth adjustment.
- ***Substitution Effect***: measures how the quantity demanded is affected by a change in the price of good l, after the wealth adjustment.
  - That is, the substitution effect only captures the change in demand due to variation in the price ratio, but abstracts from the larger (smaller) purchasing power that the consumer experiences after a decrease (increase, respectively) in prices.

Implications of WARP: Slutsky Equation

- Reduction in the price of x_1.
  - It enlarges consumer's set of feasible bundles.
  - He can reach an indifference curve further away from the origin.

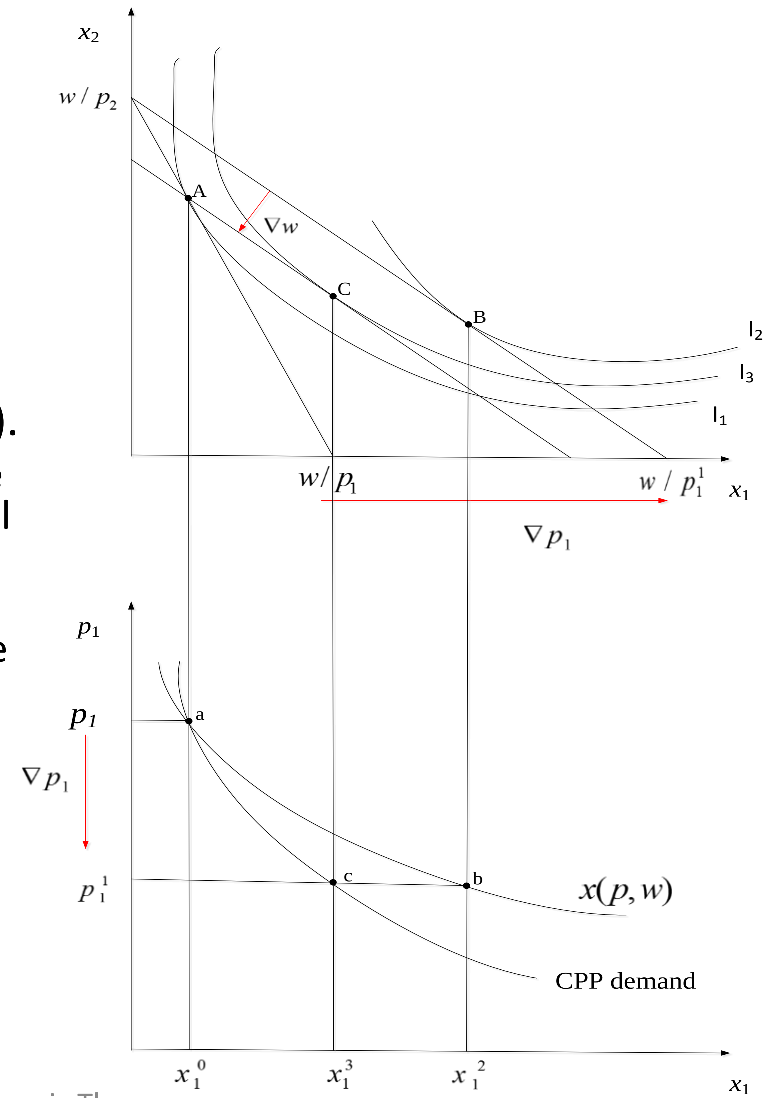
- The Walrasian demand curve indicates that a decrease in the price of x_1 leads to an increase in the quantity demanded.
  - This induces a negatively sloped Walrasian demand curve (so the good is "normal").

- The increase in the quantity demanded of x_1 as a result of a decrease in its price represents the ***total effect (TE)***.



Implications of WARP: Slutsky Equation

- Reduction in the price of x_1.
  - Disentangle the total effect into the substitution and income effects
  - ***Slutsky wealth compensation?***
- Reduce the consumer's wealth so that he can afford the same consumption bundle as the one before the price change (i.e., *A*).
  - Shift the budget line after the price change inwards until it "crosses" through the initial bundle *A*.
  - "Constant purchasing power" demand curve (CPP curve) results from applying the Slutsky wealth compensation.
  - The quantity demanded for x_1 increases from x_1^0 to x_1^3.
- When we do not hold the consumer's purchasing power constant, we observe relatively large increase in the quantity demanded for x_1 (i.e., from x_1^0 to x_1^2).

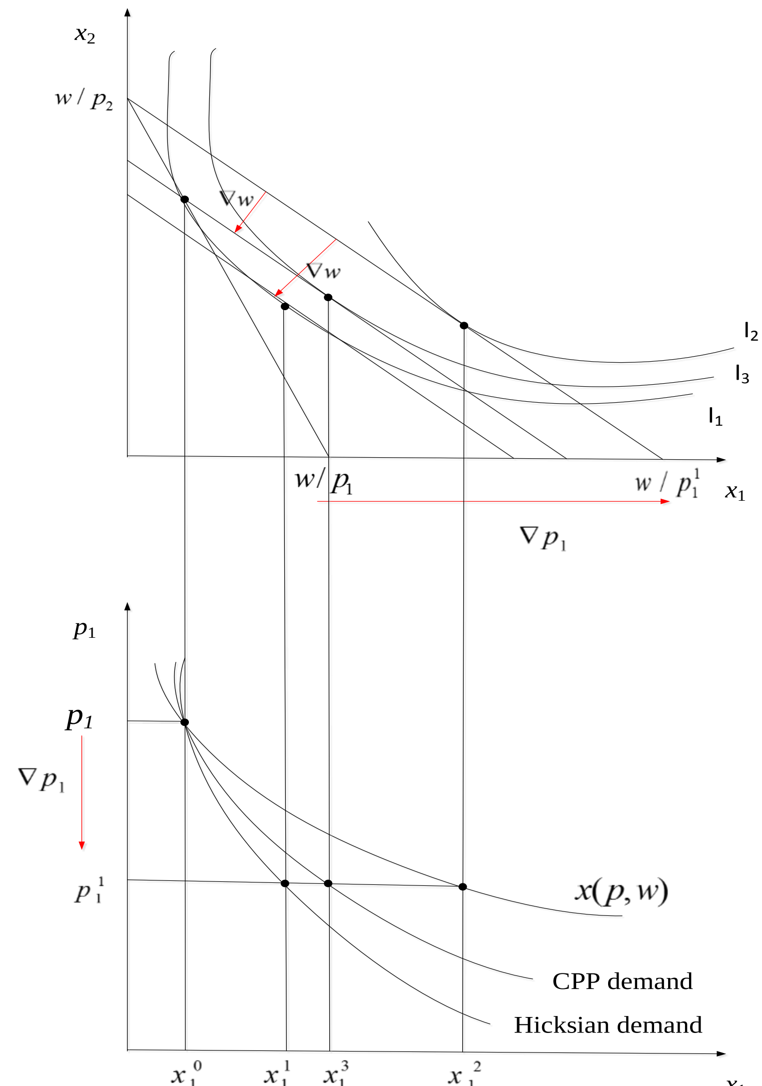


Implications of WARP: Slutsky Equation

- Reduction in the price of x_1.
  - ***Hicksian wealth compensation*** (i.e., "constant utility" demand curve)?

- The consumer's wealth level is adjusted so that he can still reach his initial utility level (i.e., the same indifference curve I_1 as before the price change).
  - A more significant wealth reduction than when we apply the Slutsky wealth compensation.
  - The Hicksian demand curve reflects that, for a given decrease in p_1, the consumer slightly increases his consumption of good one.

- In summary, a given decrease in p_1 produces:
  - A small increase in the Hicksian demand for the good, i.e., from x_1^0 to x_1^1.
  - A larger increase in the CPP demand for the good, i.e., from x_1^0 to x_1^3.
  - A substantial increase in the Walrasian demand for the product, i.e., from x_1^0 to x_1^2.

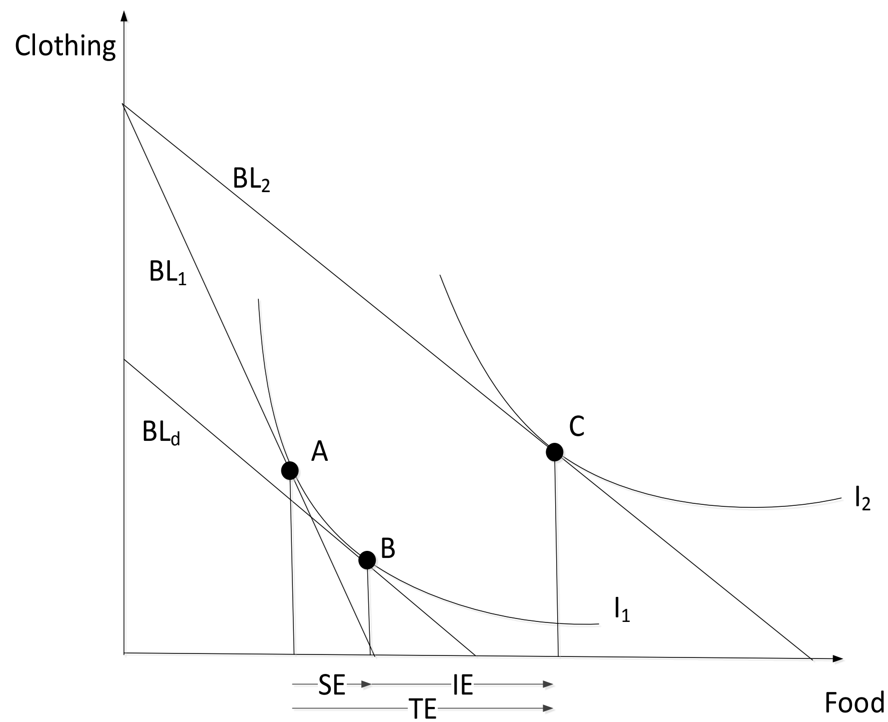


Implications of WARP: Slutsky Equation

- A decrease in price of x_1 leads the consumer to increase his consumption of this good, Δx_1, but:
  - The Δx_1 which is solely due to the price effect (either measured by the Hicksian demand curve or the CPP demand curve) is smaller than the Δx_1 measured by the Walrasian demand, $x(p, w)$, which also captures wealth effects.
  - The wealth compensation (a reduction in the consumer's wealth in this case) that maintains the original utility level (as required by the Hicksian demand) is larger than the wealth compensation that maintains his purchasing power unaltered (as required by the Slutsky wealth compensation, in the CPP curve).

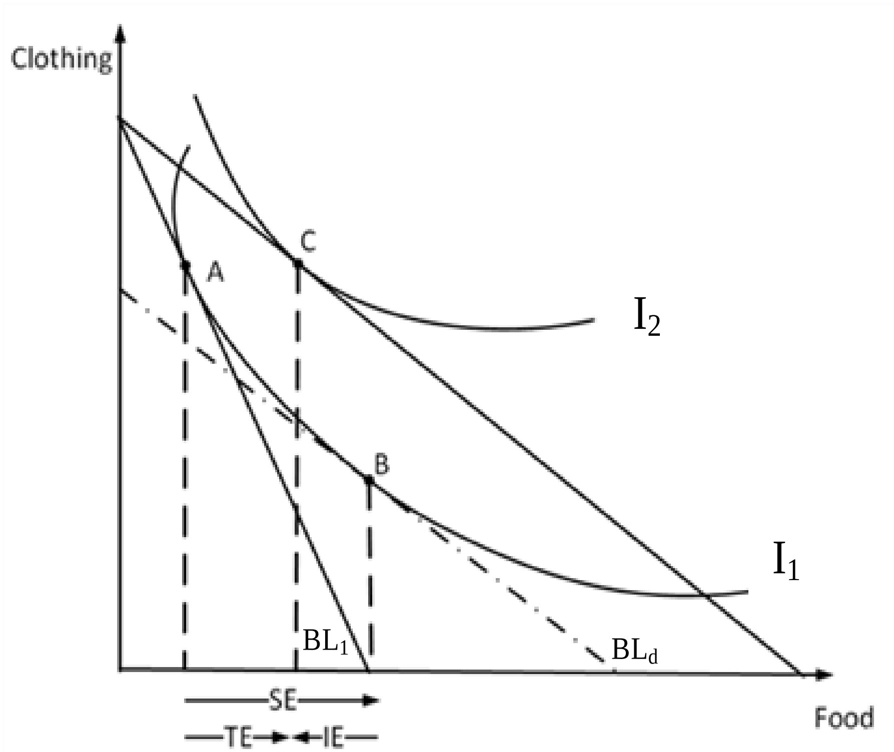
Substitution and Income Effects: Normal Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The substitution effect (SE) moves in the opposite direction as the price change.
  - A reduction in the price of food implies a positive substitution effect.
- The income effect (IE) is positive (thus it reinforces the SE).
  - The good is normal.



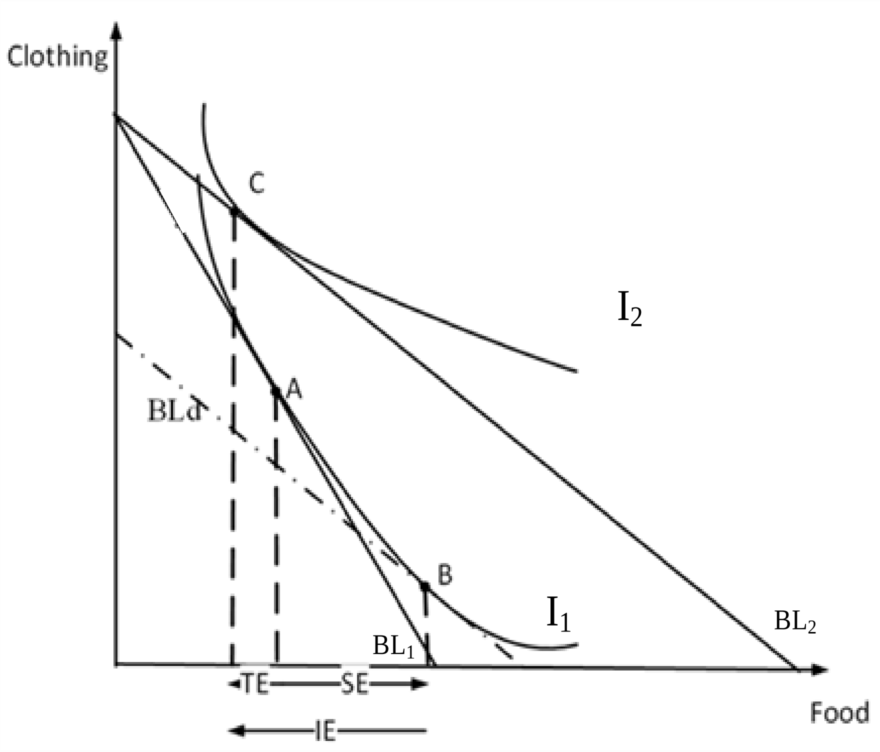
Substitution and Income Effects: Inferior Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is now negative (which partially offsets the increase in the quantity demanded associated with the SE).
  - The good is inferior.
- *Note*: the SE is larger than the IE.



Substitution and Income Effects: Giffen Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is still negative but now completely offsets the increase in the quantity demanded associated with the SE.
  - The good is Giffen good.
- *Note*: the SE is less than the IE.



Substitution and Income Effects

	SE	IE	TE
Normal Good	+	+	+
Inferior Good	+	-	+
Giffen Good	+	-	-

- Not Giffen: Demand curve is negatively sloped (as usual)
- Giffen: Demand curve is positively sloped

Substitution and Income Effects

- ***Summary:***
  1) SE is negative (since $\downarrow p_1 \Rightarrow \uparrow x_1$)
     - SE < 0 does not imply $\downarrow x_1$
  2) If good is inferior, IE < 0. Then,

$$\text{TE} = \underbrace{\text{SE}}_{-} - \underbrace{\underbrace{\text{IE}}_{-}}_{+} \quad \Rightarrow \quad \text{if } |\text{IE}| \begin{Bmatrix} > \\ < \end{Bmatrix} |\text{SE}|\text{, then } \begin{Bmatrix} \text{TE}(-) \\ \text{TE}(+) \end{Bmatrix}$$

For a price decrease, this implies

$$\begin{Bmatrix} \text{TE}(-) \\ \text{TE}(+) \end{Bmatrix} \Rightarrow \begin{Bmatrix} \downarrow x_1 \\ \uparrow x_1 \end{Bmatrix} \quad \begin{matrix} \text{Giffen good} \\ \text{Non-Giffen good} \end{matrix}$$

  3) Hence,
     a) A good can be inferior, but not necessarily be Giffen
     b) But all Giffen goods must be inferior.

Expenditure Minimization Problem

Expenditure Minimization Problem

- Expenditure minimization problem (EMP):

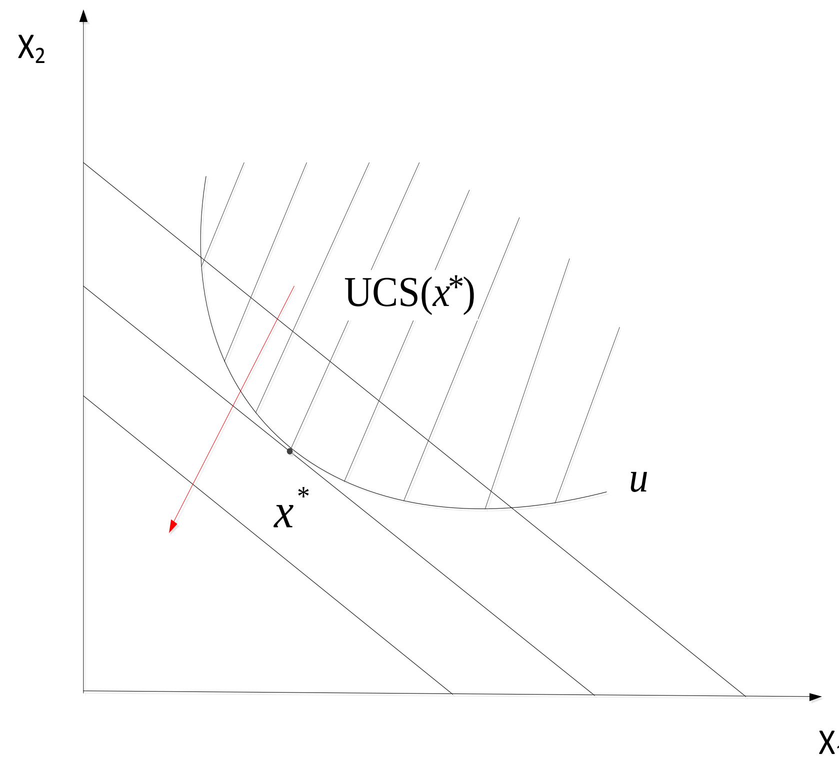
$$\min_{x \geq 0} p \cdot x$$

$$\text{s.t. } \ u(x) \geq u$$

- Alternative to utility maximization problem

Expenditure Minimization Problem

- Consumer seeks a utility level associated with a particular indifference curve, while spending as little as possible.
- Bundles strictly above x^* cannot be a solution to the EMP:
  - They reach the utility level u
  - But, they do not minimize total expenditure
- Bundles on the budget line strictly below x^* cannot be the solution to the EMP problem:
  - They are cheaper than x^*
  - But, they do not reach the utility level u



Expenditure Minimization Problem

- Lagrangian

$$L = p \cdot x + \mu[u - u(x)]$$

- FOCs (necessary conditions)

$$\frac{\partial L}{\partial x_k} = p_k - \mu \frac{\partial u(x^*)}{\partial x_k} \leq 0$$

[= 0 for interior solutions]

$$\frac{\partial L}{\partial \mu} = u - u(x^*) = 0$$

Expenditure Minimization Problem

- For interior solutions,

$$p_k = \mu \frac{\partial u(x^*)}{\partial x_k} \quad \text{or} \quad \frac{1}{\mu} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

for any good k. This implies,

$$\frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k} = \frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} \quad \text{or} \quad \frac{p_k}{p_l} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{\frac{\partial u(x^*)}{\partial x_l}}$$

- The consumer allocates his consumption across goods until the point in which the marginal utility per dollar spent on each good is equal across all goods (i.e., same "bang for the buck").

- That is, the slope of indifference curve is equal to the slope of the budget line.

EMP: Hicksian Demand

- The bundle $x^* \in \text{argmin}\, p \cdot x$ (the argument that solves the EMP) is the ***Hicksian demand***, which depends on p and u,

$$x^* \in h(p,u)$$

- Recall that if such bundle x^* is unique, we denote it as $x^* = h(p,u)$.

Properties of Hicksian Demand

- Suppose that $u(\cdot)$ is a continuous function, satisfying LNS defined on $X = \mathbb{R}^L_+$. Then for $p \gg 0$, $h(p, u)$ satisfies:
  1) ***Homog(0)*** in p, i.e., $h(p, u) = h(\alpha p, u)$ for any p, u, and $\alpha > 0$.
     - If $x^* \in h(p, u)$ is a solution to the problem
       $$\min_{x \geq 0} p \cdot x$$
       then it is also a solution to the problem
       $$\min_{x \geq 0} \alpha p \cdot x$$
     - *Intuition*: a common change in all prices does not alter the slope of the consumer's budget line.

Properties of Hicksian Demand

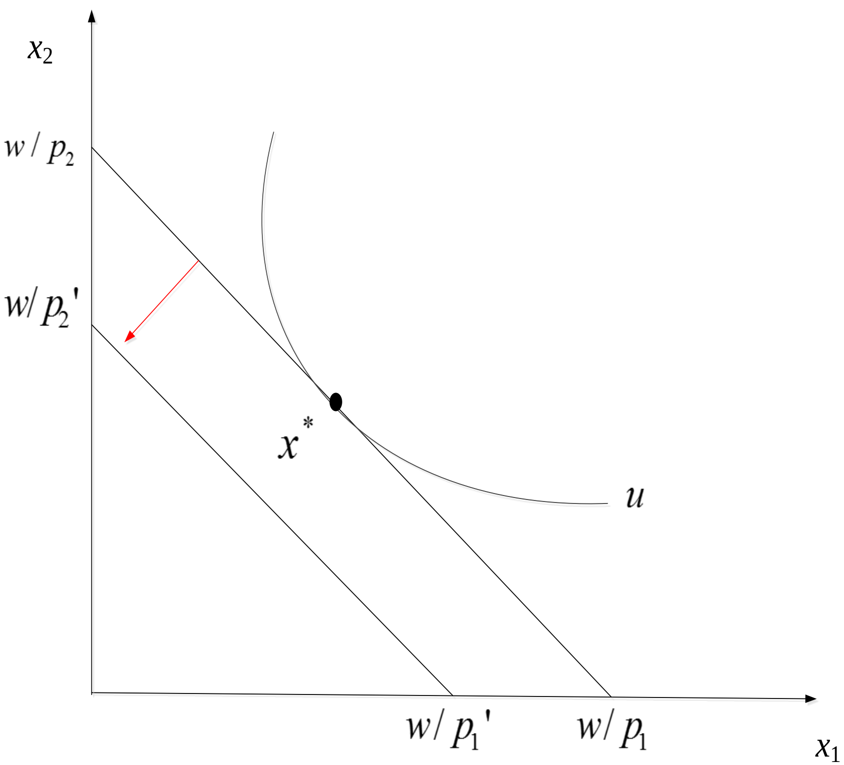
- x^* is a solution to the EMP when the price vector is $p = (p_1, p_2)$.
- Increase all prices by factor α

$$p' = (p_1', p_2') = (\alpha p_1, \alpha p_2)$$

- Downward (parallel) shift in the budget line, i.e., the slope of the budget line is unchanged.
- But I have to reach utility level u to satisfy the constraint of the EMP!
- Spend more to buy bundle $x^*(x_1^*, x_2^*)$, i.e.,

$$p_1' x_1^* + p_2' x_2^* > p_1 x_1^* + p_2 x_2^*$$

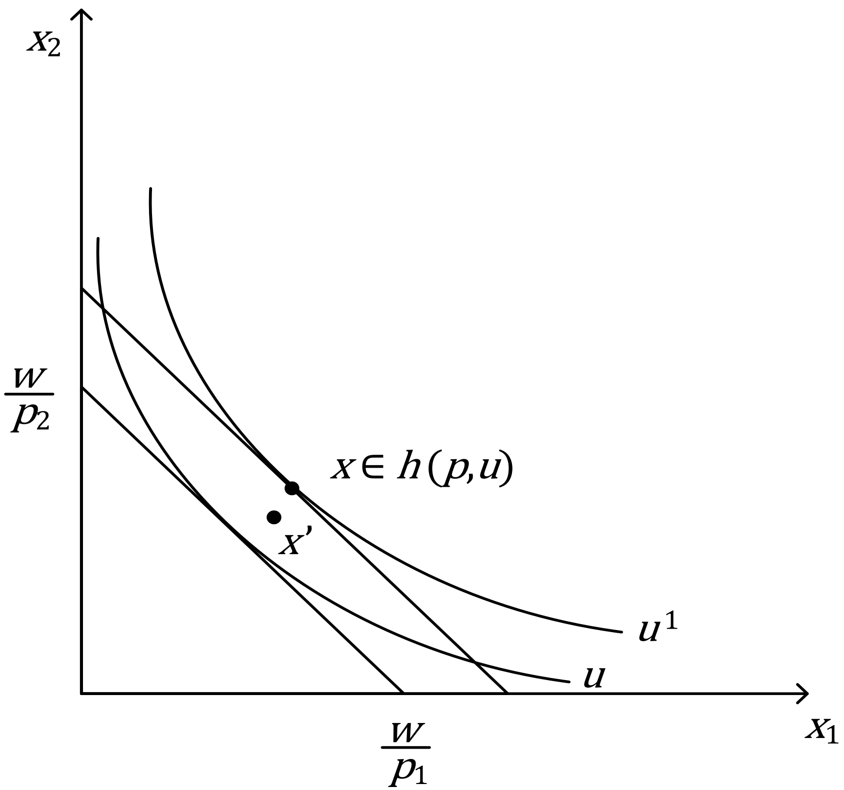
- Hence, $h(p, u) = h(\alpha p, u)$



Properties of Hicksian Demand

2) *No excess utility:*

for any optimal consumption bundle $x \in h(p, u)$, utility level satisfies $u(x) = u$.



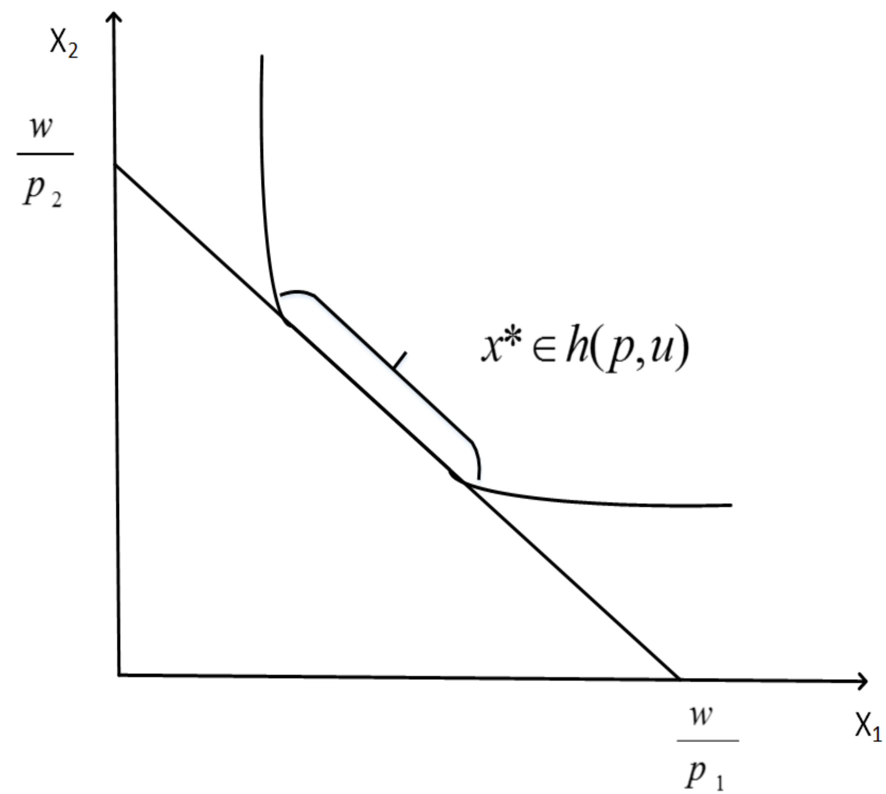
Properties of Hicksian Demand

- *Intuition*: Suppose there exists a bundle $x \in h(p,u)$ for which the consumer obtains a utility level $u(x) = u^1 > u$, which is higher than the utility level u he must reach when solving EMP.
- But we can then find another bundle $x' = x\alpha$, where $\alpha \in (0,1)$, very close to x ($\alpha \to 1$), for which $u(x') > u$.
- Bundle x':
  - is cheaper than x since it contains fewer units of all goods; and
  - exceeds the minimal utility level u that the consumer must reach in his EMP.
- We can repeat that argument until reaching bundle x.
- In summary, for a given utility level u that you seek to reach in the EMP, bundle $h(p,u)$ does not exceed u. Otherwise you can find a cheaper bundle that exactly reaches u.

Properties of Hicksian Demand

3) *Convexity:*

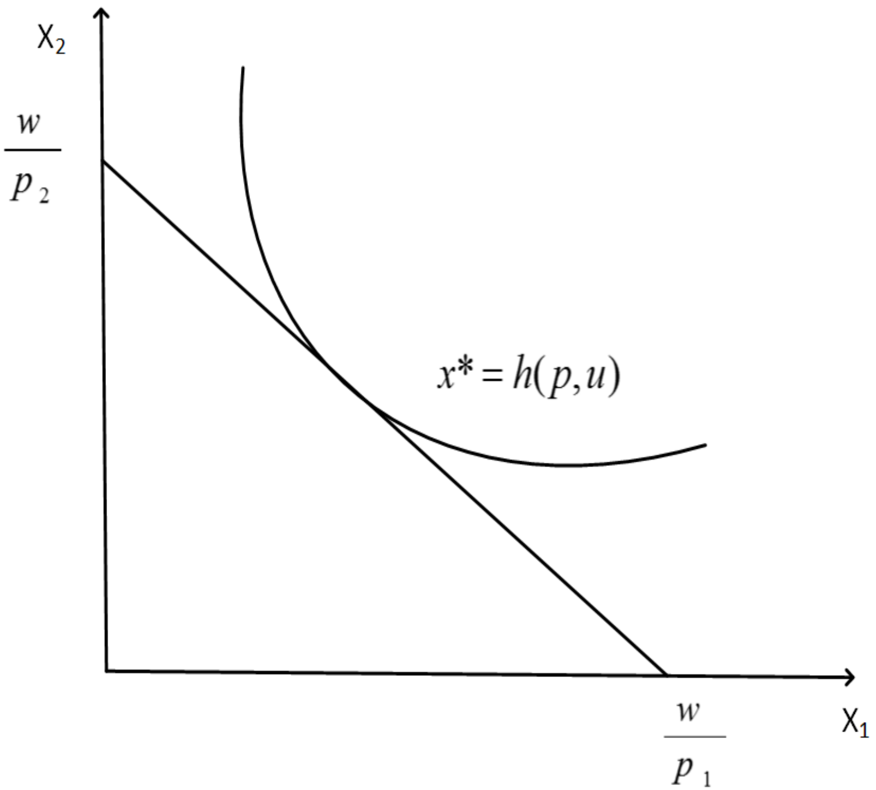
If the preference relation is convex, then $h(p,u)$ is a convex set.



Properties of Hicksian Demand

4) Uniqueness:

If the preference relation is strictly convex, then $h(p, u)$ contains a single element.



Properties of Hicksian Demand

- ***Compensated Law of Demand:*** for any change in prices p and p',

$$(p'-p)\cdot[h(p',u)-h(p,u)]\leq 0$$

  - *Implication*: for every good k,

$$(p'_k-p_k)\cdot[h_k(p',u)-h_k(p,u)]\leq 0$$

  - This is true for compensated demand, but not necessarily true for Walrasian demand (which is uncompensated):
    - Recall the figures on Giffen goods, where a decrease in p_k in fact decreases $x_k(p,u)$ when wealth was left uncompensated.

The Expenditure Function

- Plugging the result from the EMP, $h(p, u)$, into the objective function, $p \cdot x$, we obtain the value function of this optimization problem,

$$p \cdot h(p, u) = e(p, u)$$

where $e(p, u)$ represents the ***minimal expenditure*** that the consumer needs to incur in order to reach utility level u when prices are p.

Properties of Expenditure Function

- Suppose that $u(\cdot)$ is a continuous function, satisfying LNS defined on $X = \mathbb{R}^L_+$. Then for $p \gg 0$, $e(p, u)$ satisfies:

  1) ***Homog(1)*** in p,

$$e(\alpha p, u) = \alpha \underbrace{p \cdot x^*}_{e(p,u)} = \alpha \cdot e(p, u)$$

for any p, u, and $\alpha > 0$.

  - We know that the optimal bundle is not changed when all prices change, since the optimal consumption bundle in $h(p, u)$ satisfies homogeneity of degree zero.
  - Such a price change just makes it more or less expensive to buy the same bundle.

Properties of Expenditure Function

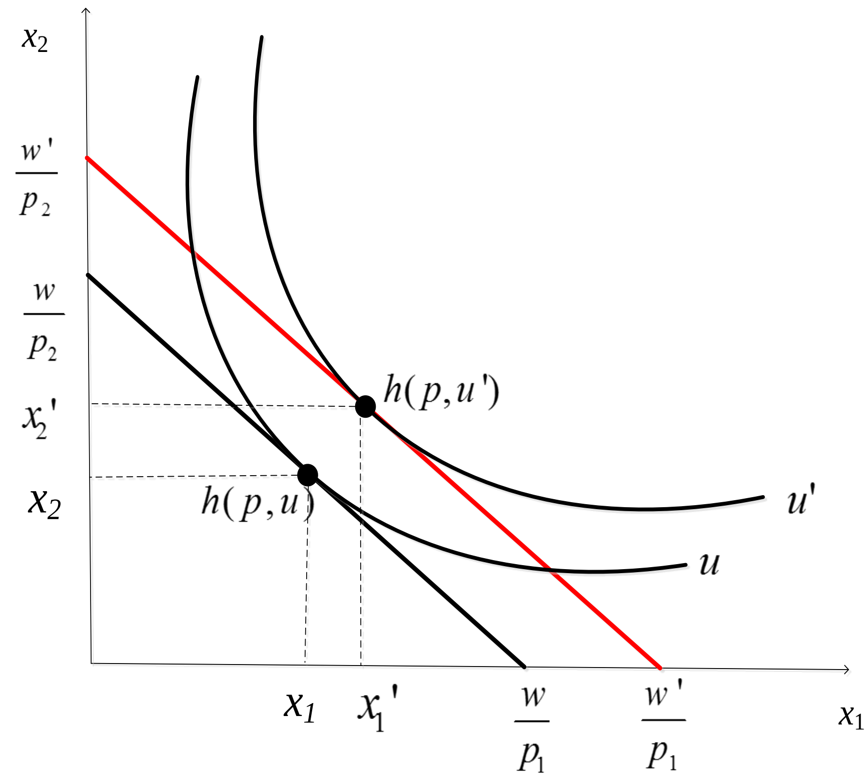
2) Strictly increasing in u:
For a given price vector, reaching a higher utility requires higher expenditure:

$$p_1 x_1' + p_2 x_2' > p_1 x_1 + p_2 x_2$$

where $(x_1, x_2) = h(p, u)$ and $(x_1', x_2') = h(p, u')$.

Then,

$$e(p, u') > e(p, u)$$



Properties of Expenditure Function

3) ***Non-decreasing in p_k for every good k:***

Higher prices mean higher expenditure to reach a given utility level.

- Let $p' = (p_1, p_2, \dots, p_k', \dots, p_L)$ and $p = (p_1, p_2, \dots, p_k, \dots, p_L)$, where $p_k' > p_k$.
- Let $x' = h(p', u)$ and $x = h(p, u)$ from EMP under prices p' and p, respectively.
- Then, $p' \cdot x' = e(p', u)$ and $p \cdot x = e(p, u)$.

$$e(p', u) = p' \cdot x' \geq p \cdot x' \geq p \cdot x = e(p, u)$$

- 1st inequality due to $p' \geq p$
- 2nd inequality: at prices p, bundle x minimizes EMP.

Properties of Expenditure Function

4) Concave in p:

Let $x' \in h(p', u) \Rightarrow p'x' \leq p'x$
$\forall x \neq x'$, e.g., $p'x' \leq p'\bar{x}$
and
$x'' \in h(p'', u) \Rightarrow p''x'' \leq p''x$
$\forall x \neq x''$, e.g., $p''x'' \leq p''\bar{x}$
where $\bar{x} = \alpha x' + (1-\alpha)x''$

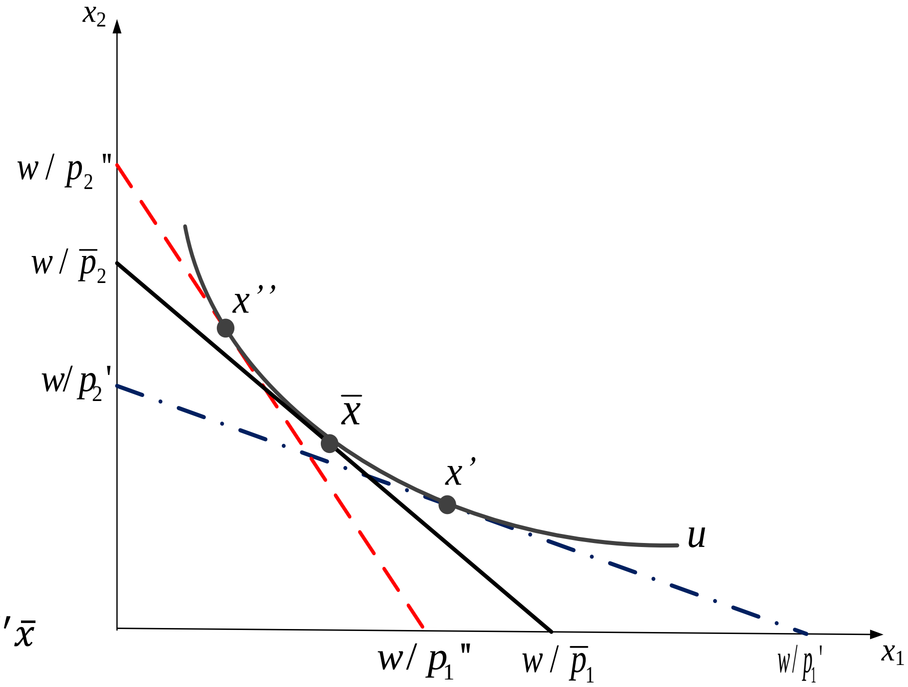
This implies

$$\alpha p'x' + (1-\alpha)p''x'' \leq \alpha p'\bar{x} + (1-\alpha)p''\bar{x}$$

$$\alpha \overbrace{p'x'}^{e(p',u)} + (1-\alpha) \overbrace{p''x''}^{e(p'',u)} \leq [\underbrace{\alpha p' + (1-\alpha)p''}_{\bar{p}}]\bar{x}$$

$$\alpha e(p', u) + (1-\alpha)e(p'', u) \leq e(\bar{p}, u)$$

as required by concavity



Connections

Relationship between the Expenditure and Hicksian Demand

- Let's assume that $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}_+^L$. For all p and u,

$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u) \text{ for every good } k$$

  This identity is "***Shepard's lemma***": if we want to find $h_k(p,u)$ and we know $e(p,u)$, we just have to differentiate $e(p,u)$ with respect to prices.

- *Proof*: three different approaches
  1) the support function
  2) first-order conditions
  3) the envelope theorem (See Appendix 2.2)

Relationship between the Expenditure and Hicksian Demand

- The relationship between the Hicksian demand and the expenditure function can be further developed by taking first order conditions again. That is,

$$\frac{\partial^2 e(p,u)}{\partial p_k^2} = \frac{\partial h_k(p,u)}{\partial p_k}$$

or

$$D_p^2 e(p,u) = D_p h(p,u)$$

- Since $D_p h(p,u)$ provides the Slutsky matrix, $S(p,w)$, then

$$S(p,w) = D_p^2 e(p,u)$$

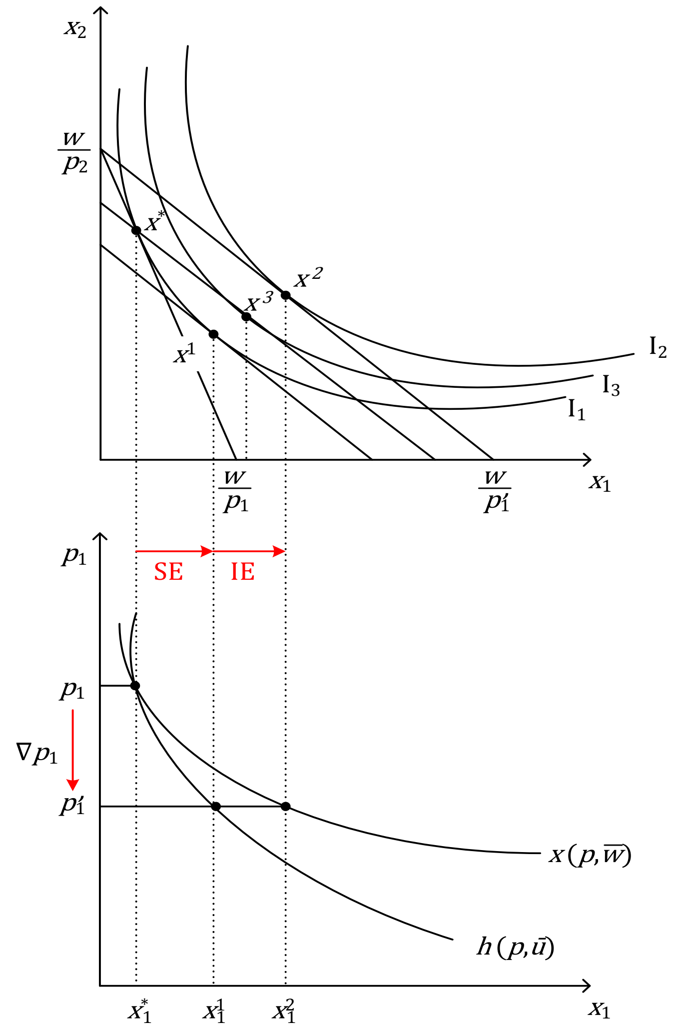
where the Slutsky matrix can be obtained from the observable Walrasian demand.

Relationship between the Expenditure and Hicksian Demand

- There are three other important properties of $D_p h(p,u)$, where $D_p h(p,u)$ is $L \times L$ derivative matrix of the hicksian demand, $h(p,u)$:
  1) $D_p h(p,u)$ is negative semidefinite
     - Hence, $S(p,w)$ is negative semidefinite.
  2) $D_p h(p,u)$ is a symmetric matrix
     - Hence, $S(p,w)$ is symmetric.
  3) $D_p h(p,u)p = 0$, which implies $S(p,w)p = 0$.
     - Not all goods can be net substitutes $\left(\frac{\partial h_l(p,u)}{\partial p_k} > 0\right)$ or net complements $\left(\frac{\partial h_l(p,u)}{\partial p_k} < 0\right)$. Otherwise, the multiplication of this vector of derivatives times the (positive) price vector $p \gg 0$ would yield a non-zero result.

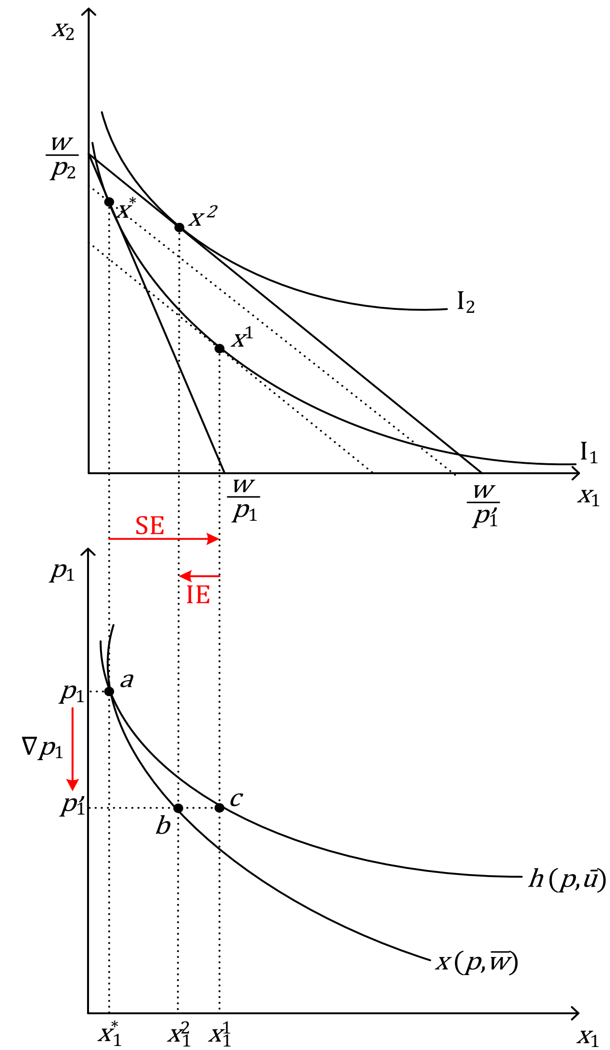
Relationship between Hicksian and Walrasian Demand

- When income effects are positive (*normal goods*), then the Walrasian demand $x(p, w)$ is **above** the Hicksian demand $h(p, u)$.
  – The Hicksian demand is *steeper* than the Walrasian demand.



Relationship between Hicksian and Walrasian Demand

- When income effects are negative (*inferior goods*), then the Walrasian demand $x(p, w)$ is **below** the Hicksian demand $h(p, u)$.
  - The Hicksian demand is *flatter* than the Walrasian demand.



Relationship between Hicksian and Walrasian Demand

- We can formally relate the Hicksian and Walrasian demand as follows:
  - Consider $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}_+^L$.
  - Consider a consumer facing $(\bar{p}, \bar{w})$ and attaining utility level $\bar{u}$.
  - Note that $\bar{w} = e(\bar{p}, \bar{u})$. In addition, we know that for any (p, u), $h_l(p, u) = x_l(p, \underbrace{e(p, u)}_{w})$. Differentiating this expression with respect to p_k, and evaluating it at $(\bar{p}, \bar{u})$, we get:

$$\frac{\partial h_l(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} \frac{\partial e(\bar{p}, \bar{u})}{\partial p_k}$$

Relationship between Hicksian and Walrasian Demand

- Using the fact that $\frac{\partial e(\bar{p},\bar{u})}{\partial p_k} = h_k(\bar{p},\bar{u})$,

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial w} h_k(\bar{p},\bar{u})$$

- Finally, since $\bar{w} = e(\bar{p},\bar{u})$ and $h_k(\bar{p},\bar{u}) = x_k\big(\bar{p},e(\bar{p},\bar{u})\big) = x_k(\bar{p},\bar{w})$, then

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},\bar{w})}{\partial p_k} + \frac{\partial x_l(\bar{p},\bar{w})}{\partial w} x_k(\bar{p},\bar{w})$$

Relationship between Hicksian and Walrasian Demand

- But this coincides with $s_{lk}(p, w)$ that we discussed in the Slutsky equation.
  - Hence, we have $\underbrace{D_p h(p, u)}_{L \times L} = S(p, w)$.
  - Or, more compactly, $SE = TE + IE$.

Relationship between Walrasian Demand and Indirect Utility Function

- Let's assume that $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}_+^L$. Suppose also that $v(p, w)$ is differentiable at any $(p, w) \gg 0$. Then,

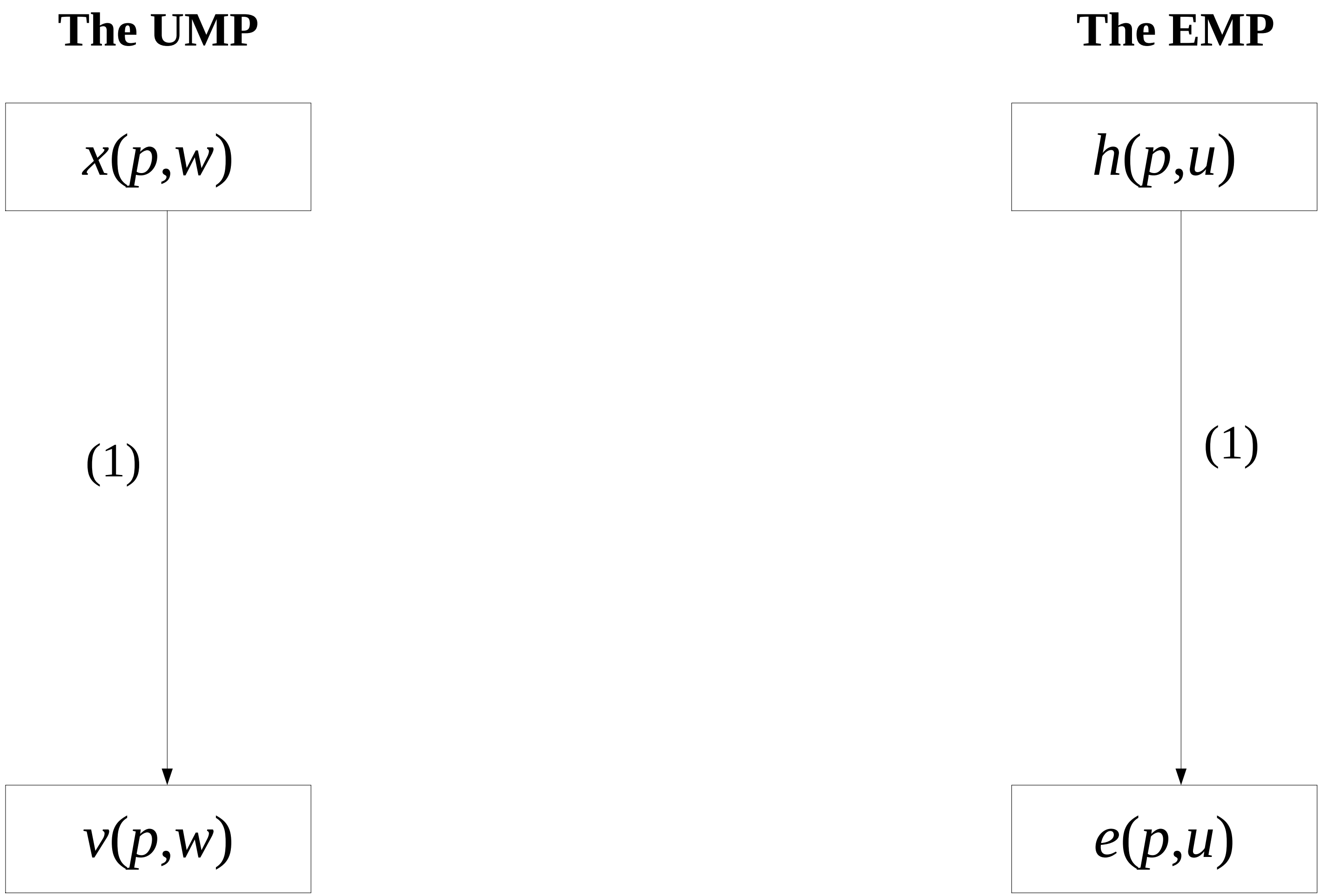
$$-\frac{\frac{\partial v(p,w)}{\partial p_k}}{\frac{\partial v(p,w)}{\partial w}} = x_k(p, w) \text{ for every good } k$$

- This is ***Roy's identity***.
- Powerful result, since in many cases it is easier to compute the derivatives of $v(p, w)$ than solving the UMP with the system of FOCs.

Summary of Relationships

- The Walrasian demand, $x(p,w)$, is the solution of the UMP.
  - Its value function is the indirect utility function, $v(p,w)$.
- The Hicksian demand, $h(p,u)$, is the solution of the EMP.
  - Its value function is the expenditure function, $e(p,u)$.

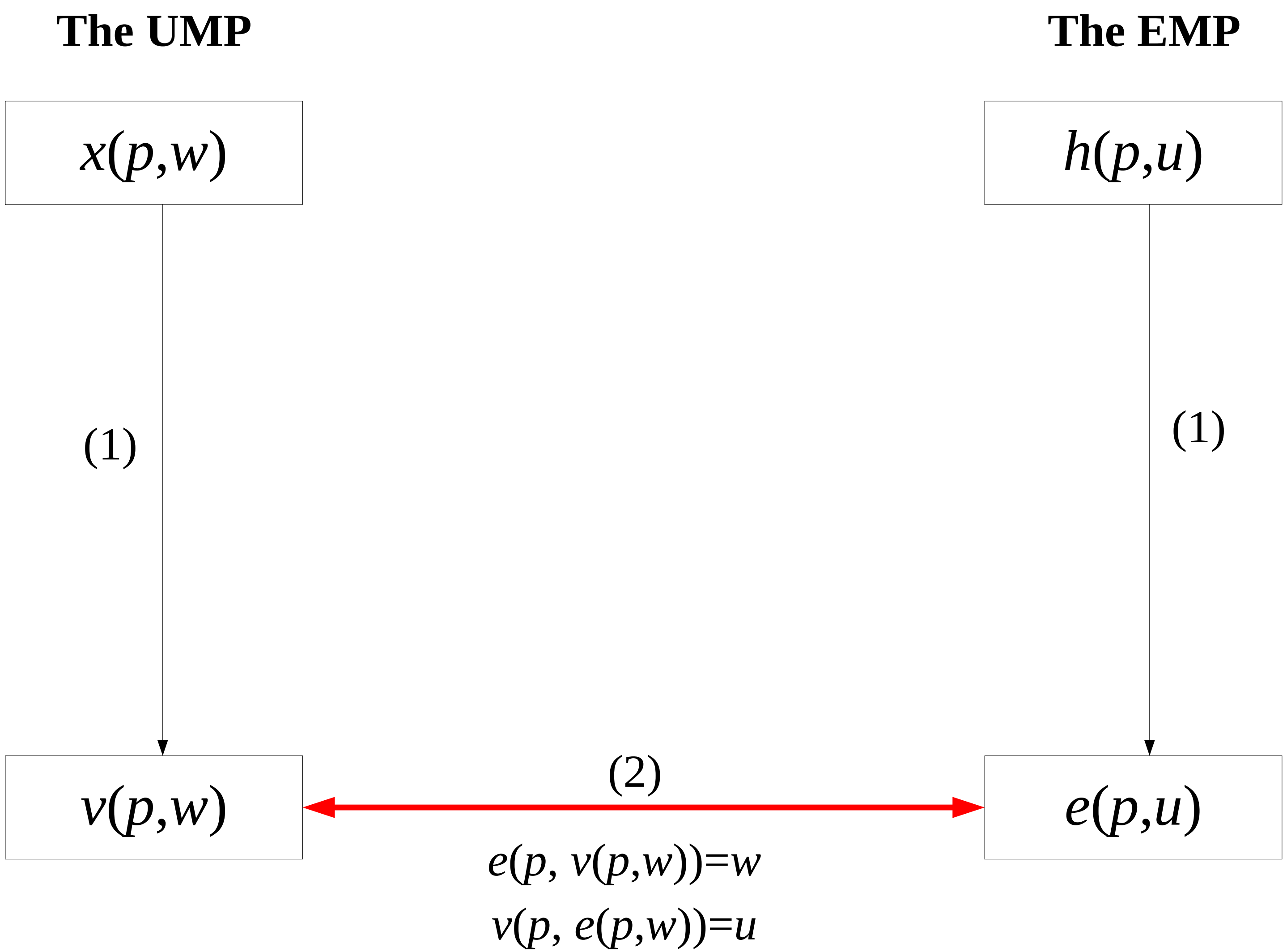
Summary of Relationships



Summary of Relationships

- Relationship between the value functions of the UMP and the EMP (lower part of figure):
  - $e(p, v(p, w)) = w$, i.e., the minimal expenditure needed in order to reach a utility level equal to the maximal utility that the individual reaches at his UMP, $u = v(p, w)$, must be w.
  - $v(p, e(p, u)) = u$, i.e., the indirect utility that can be reached when the consumer is endowed with a wealth level w equal to the minimal expenditure he optimally uses in the EMP, i.e., $w = e(p, u)$, is exactly u.

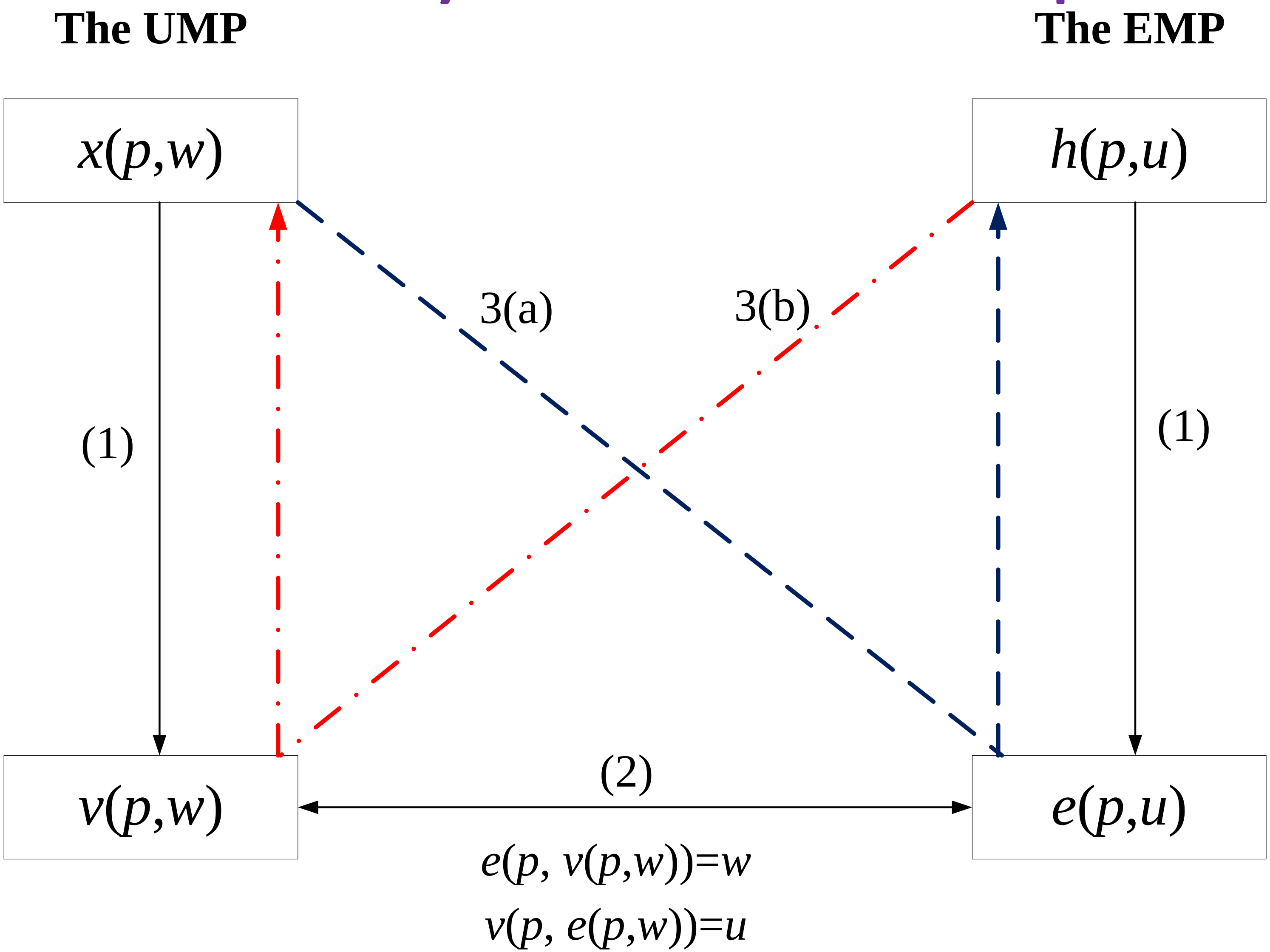
Summary of Relationships



Summary of Relationships

- Relationship between the argmax of the UMP (the Walrasian demand) and the argmin of the EMP (the Hicksian demand):
  - $x(p, e(p, u)) = h(p, u)$, i.e., the (uncompensated) Walrasian demand of a consumer endowed with an adjusted wealth level w (equal to the expenditure he optimally uses in the EMP), $w = e(p, u)$, coincides with his Hicksian demand, $h(p, u)$.
  - $h(p, v(p, w)) = x(p, w)$, i.e., the (compensated) Hicksian demand of a consumer reaching the maximum utility of the UMP, $u = v\ (p, w)$, coincides with his Walrasian demand, $x(p, w)$.

Summary of Relationships



Summary of Relationships

- Finally, we can also use:
  - The ***Slutsky equation***:

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

    to relate the derivatives of the Hicksian and the Walrasian demand.
  - ***Shepard's lemma***:

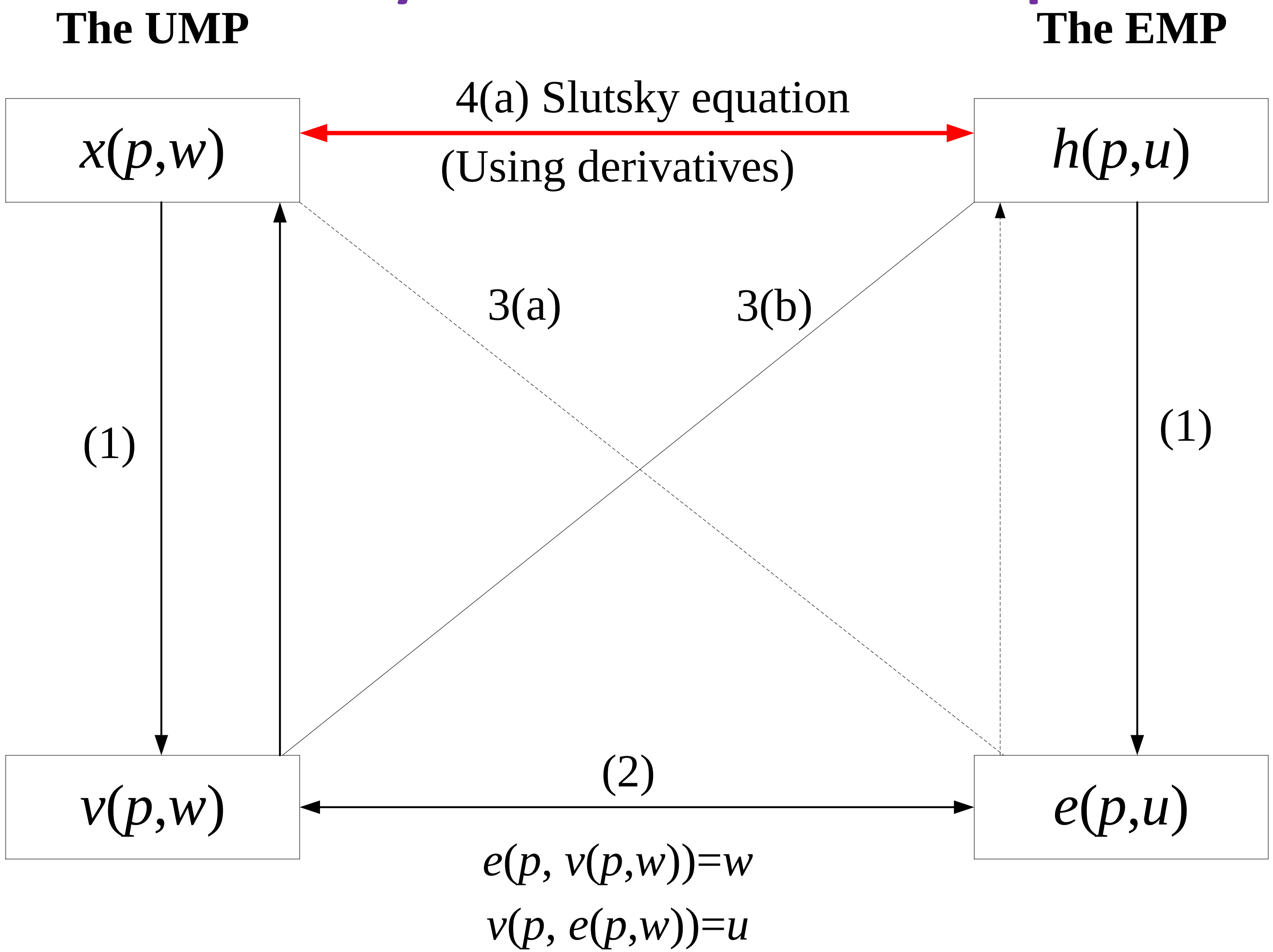
$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u)$$

    to obtain the Hicksian demand from the expenditure function.
  - ***Roy's identity***:

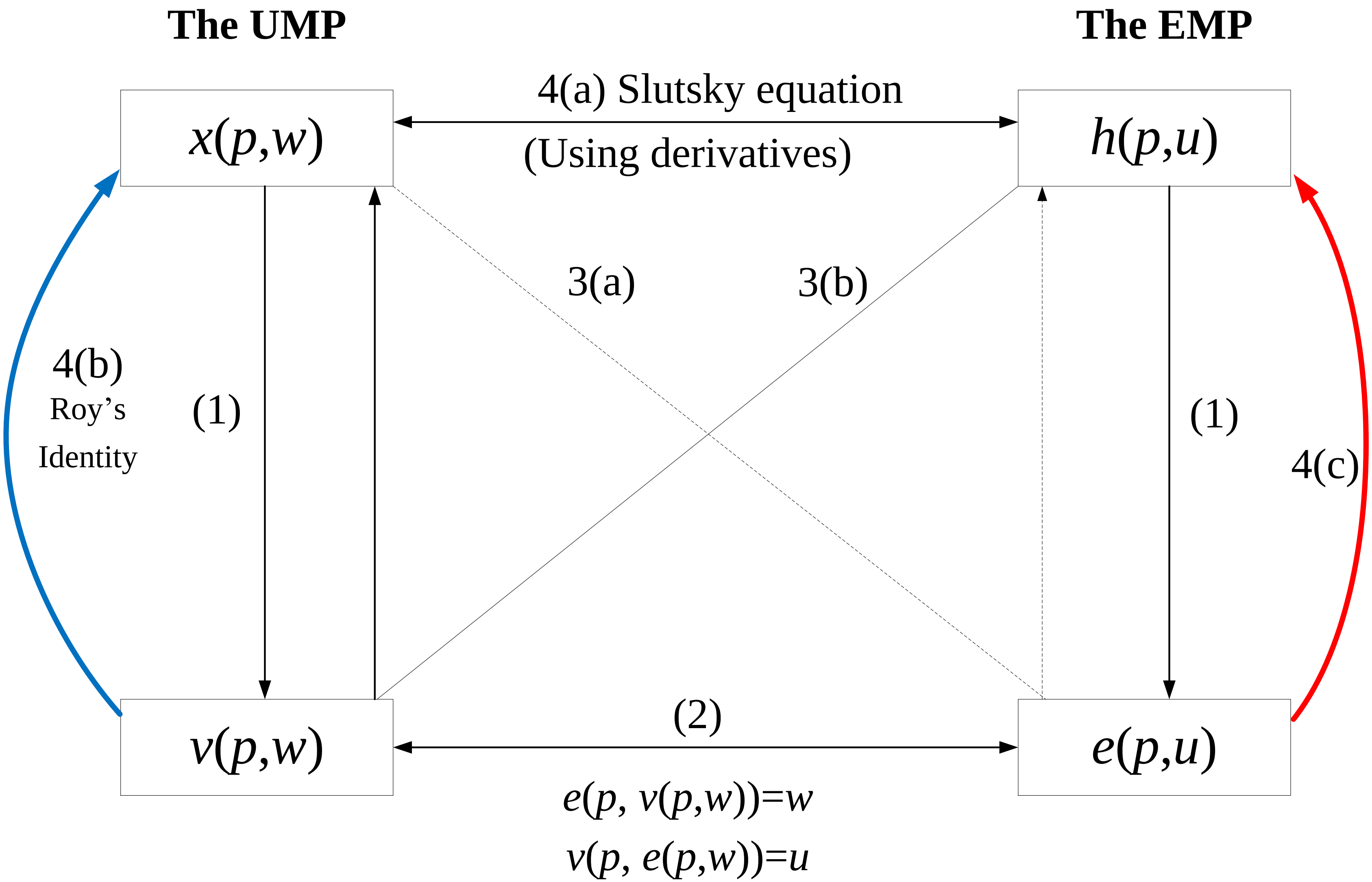
$$-\frac{\frac{\partial v(p,w)}{\partial p_k}}{\frac{\partial v(p,w)}{\partial w}} = x_k(p,w)$$

    to obtain the Walrasian demand from the indirect utility function.

Summary of Relationships



Summary of Relationships



Appendix 2.1: Duality in Consumption

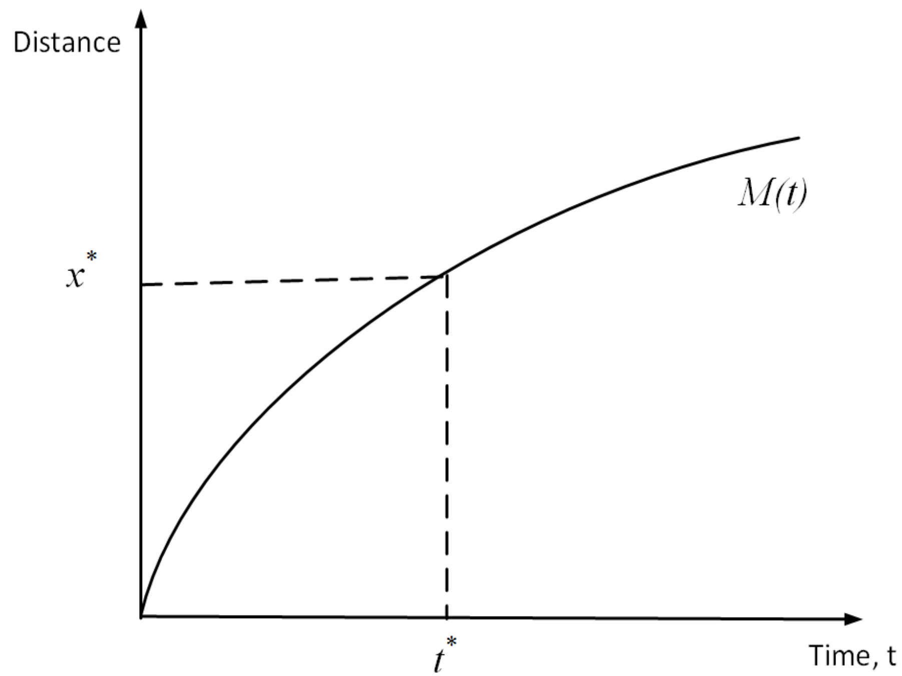
Duality in Consumption

- We discussed the utility maximization problem (UMP) – the so-called *primal* problem describing the consumer's choice of optimal consumption bundles – and its *dual*: the expenditure minimization problem (EMP).

- When can we guarantee that the solution x^* to both problems coincide?

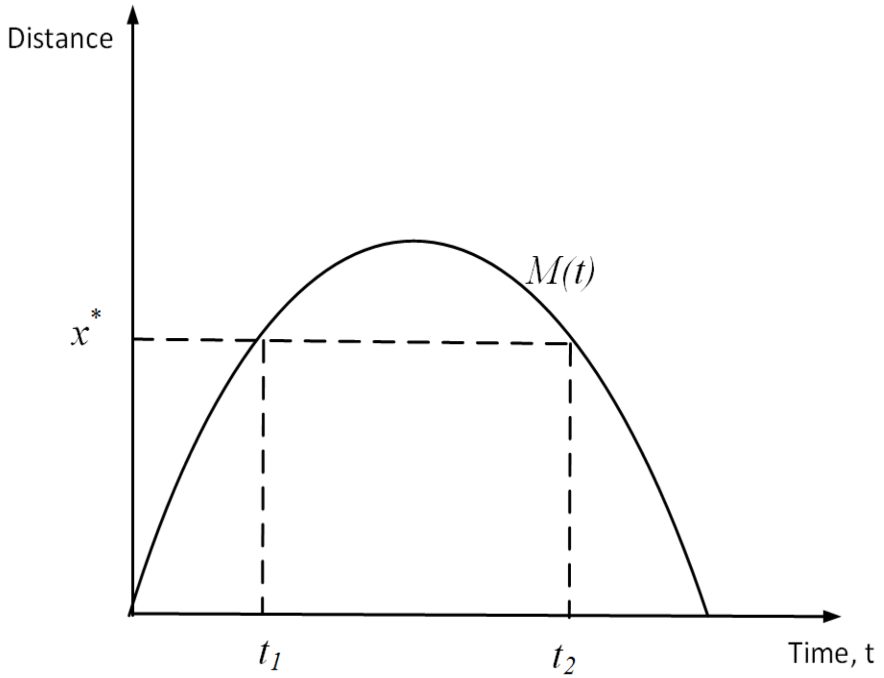
Duality in Consumption

- The maximal distance that a turtle can travel in t^* time is x^*.
  - From time (t^*) to distance (x^*)
- The minimal time that a turtle needs to travel x^* distance is t^*.
  - From distance (x^*) to time (t^*)
- In this case the primal and dual problems would provide us with the same answer to both of the above questions



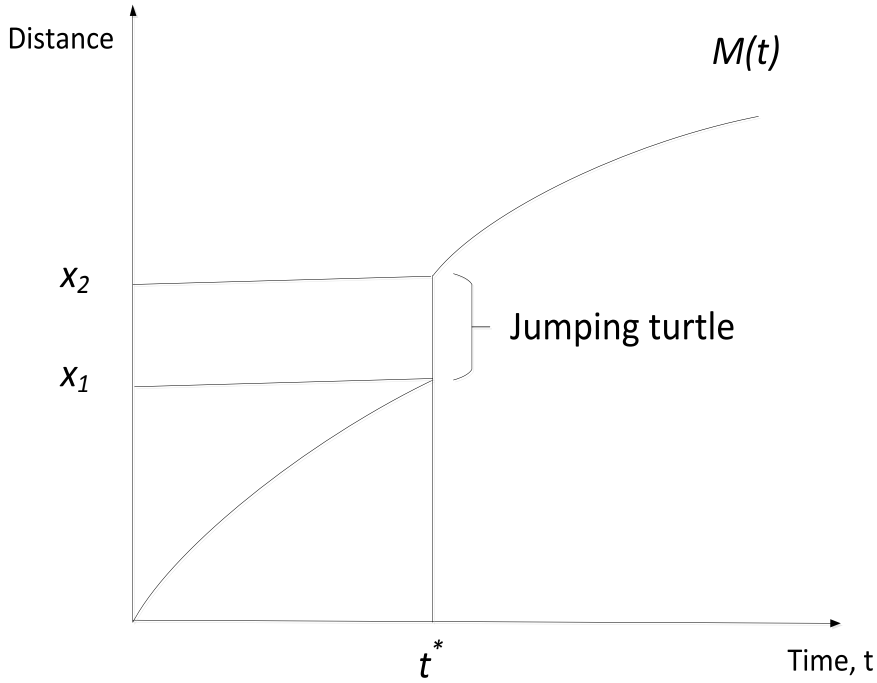
Duality in Consumption

- In order to obtain the same answer from both questions, we critically need that the function $M(t)$ satisfies ***monotonicity***.
- Otherwise: the maximal distance traveled at both t_1 and t_2 is x^*, but the minimal time that a turtle needs to travel x^* distance is t_1.
- Hence, a non-monotonic function cannot guarantee that the answers from both questions are compatible.



Duality in Consumption

- Similarly, we also require that the function $M(t)$ satisfies ***continuity***.
- Otherwise: the maximal distance that the turtle can travel in time t^* is x_2 , whereas the minimum time required to travel distance x_1 and x_2 is t^* for both distances.
- Hence, a non-continuous function does not guarantee that the answers to both questions are compatible.



Duality in Consumption

- Given $u(\cdot)$ is monotonic and continuous, then if x^* is the solution to the problem

$$\max_{x \geq 0} u(x) \quad \text{s.t. } p \cdot x \leq w \qquad \text{(UMP)}$$

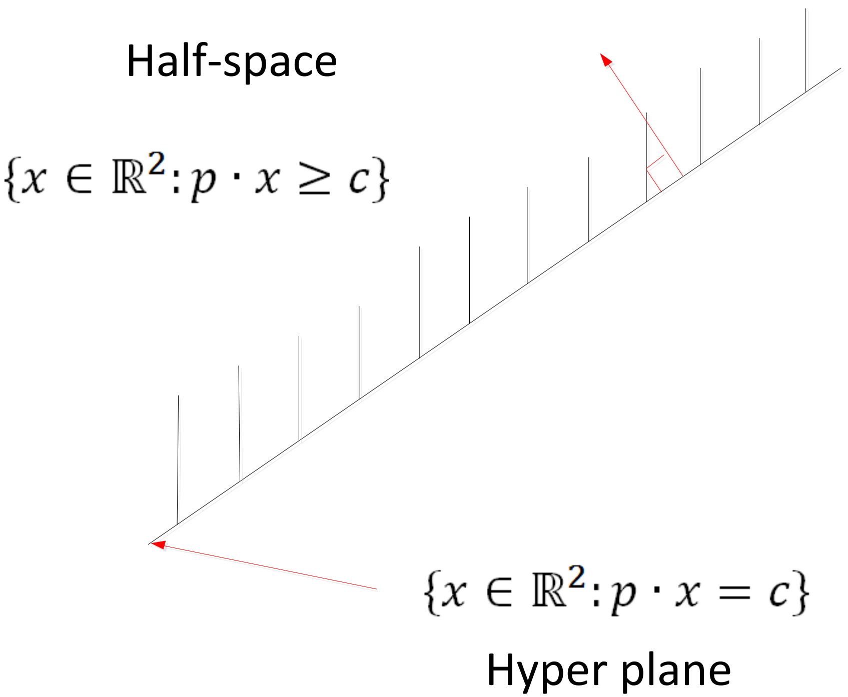
it must also be a solution to the problem

$$\min_{x \geq 0} p \cdot x \quad \text{s.t. } u(x) \geq u \qquad \text{(EMP)}$$

Hyperplane Theorem

- ***Hyperplane***: for some $p \in \mathbb{R}^L_+$ and $c \in \mathbb{R}$, the set of points in $\mathbb{R}^L$ such that
 $\{x \in \mathbb{R}^L\colon p \cdot x = c\}$

- ***Half-space***: the set of bundles x for which $p \cdot x \geq c$. That is,
 $\{x \in \mathbb{R}^L\colon p \cdot x \geq c\}$



Hyperplane Theorem

- ***Separating hyperplane theorem***: For every convex and closed set K, there is a half-space containing K and excluding any point $\bar{x} \notin K$ outside of this set.
  - That is, there exist $p \in \mathbb{R}_+^L$ and $c \in \mathbb{R}$ such that

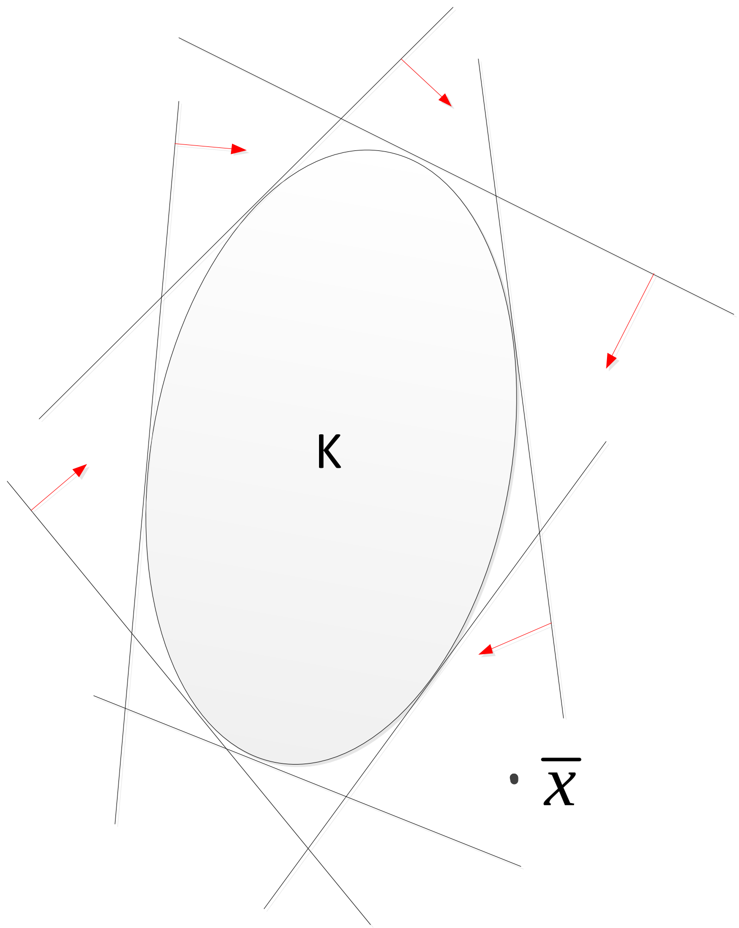
    $$p \cdot x \geq c \text{ for all elements in the set, } x \in K$$

    $$p \cdot \bar{x} < c \text{ for all elements outside the set, } \bar{x} \notin K$$

  - *Intuition*: every convex and closed set K can be equivalently described as the intersection of the half-spaces that contain it.
    - As we draw more and more half spaces, their intersection becomes the set K.

Hyperplane Theorem

- If K is a closed and convex set, we can then construct half-spaces for all the elements in the set $(x_1, x_2, \ldots)$ such that their intersections coincides ("equivalently describes") set K.
  - We construct a cage (or hull) around the convex set K that exactly coincides with set K.
- Bundle like $\bar{x} \notin K$ lies outside the intersection of half-spaces.



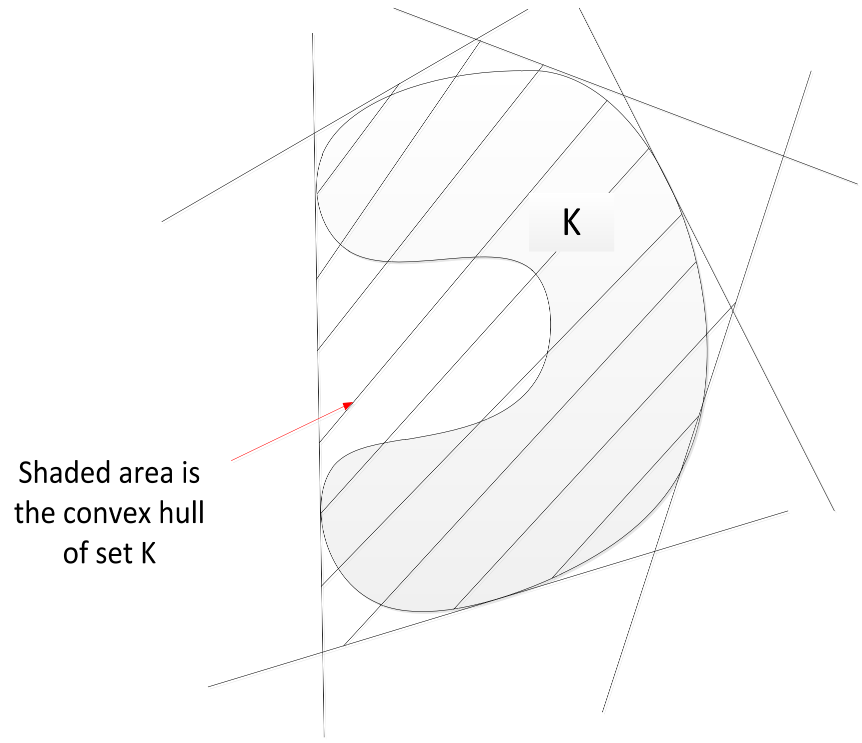
Hyperplane Theorem

- What if the set we are trying to "equivalently describe" by the use of half-spaces is *non-convex*?
  - The intersection of half-spaces does not coincide with set K (it is, in fact, larger, since it includes points outside set K). Hence, we cannot use several half-spaces to "equivalently describe" set K.
  - Then the intersection of half-spaces that contain K is the smallest, convex set that contains K, known as the closed, convex hull of K.

Hyperplane Theorem

- The convex hull of set K is often denoted as $\overline{K}$, and it is convex (unlike set K, which might not be convex).

- The convex hull $\overline{K}$ is convex, both when set K is convex and when it is not.



Support Function

- For every nonempty closed set $K \subset \mathbb{R}^L$, its ***support function*** is defined by

$$\mu_K(p) = \inf_x \{p \cdot x\} \text{ for all } x \in K \text{ and } p \in \mathbb{R}^L$$

that is, the support function finds, for a given price vector p, the bundle x that minimizes $p \cdot x$.
  - Recall that inf coincides with the argmin when the constraint includes the boundary.

- From the support function of K, we can reconstruct K.
  - In particular, for every $p \in \mathbb{R}^L$, we can define half-spaces whose boundary is the support function of set K.
    - That is, we define the set of bundles for which $p \cdot x \geq \mu_K(p)$. Note that all bundles x in such half-space contains elements in the set K, but does not contain elements outside K, i.e., $\bar{x} \notin K$.

Support Function

- Thus, the intersection of the half-spaces generated by all possible values of p describes ("reconstructs") the set K. That is, set K can be described by all those bundles $x \in \mathbb{R}^L$ such that

$$K = \{x \in \mathbb{R}^L : p \cdot x \geq \mu_K(p) \text{ for every } p\}$$

- By the same logic, if K is not convex, then the set

$$\{x \in \mathbb{R}^L : p \cdot x \geq \mu_K(p) \text{ for every } p\}$$

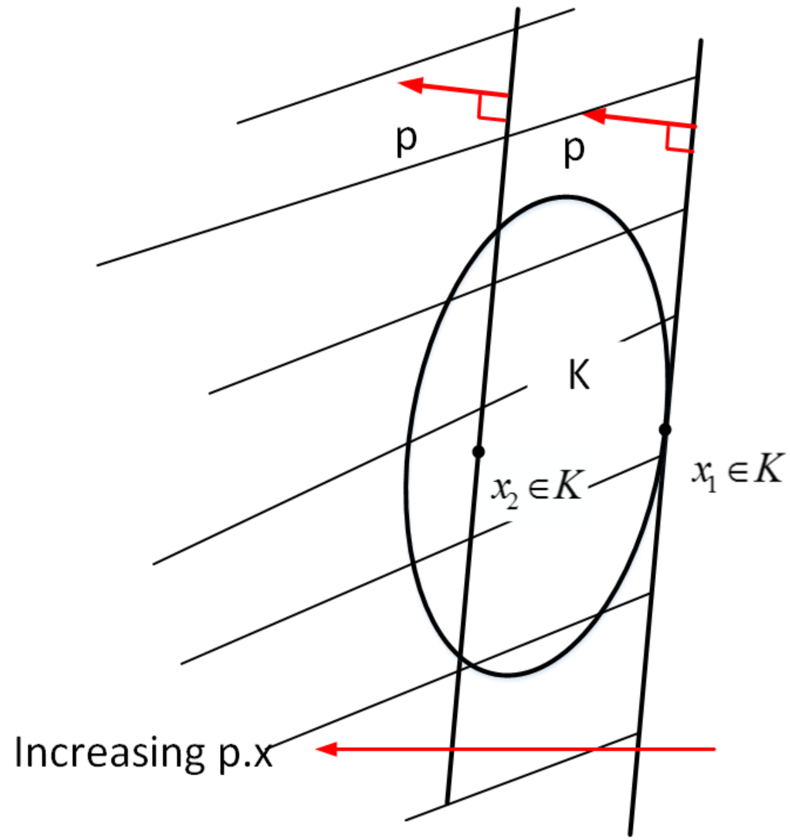
defines the smallest closed, convex set containing K (i.e., the convex hull of set K).

Support Function

- For a given p, the support $\mu_K(p)$ selects the element in K that minimizes $p \cdot x$ (i.e., x_1 in this example).
- Then, we can define the half-space of that hyperlane as:

$$p \cdot x \geq \underbrace{p \cdot x_1}_{\mu_K(p)}$$

  - The above inequality identifies all bundles x to the left of hyperplane $p \cdot x_1$.
- We can repeat the same procedure for any other price vector p.



Support Function

- The above definition of the support function provides us with a useful ***duality theorem***:
  - Let K be a nonempty, closed set, and let $\mu_K(\cdot)$ be its support function. Then there is a unique element in K, $\bar{x} \in K$, such that, for all price vector $\bar{p}$,

$$\bar{p} \cdot \bar{x} = \mu_K(\bar{p}) \iff \mu_K(\cdot) \text{ is differentiable at } \bar{p}$$

$$p \cdot h(p,u) = e(p,u) \iff e(\cdot,u) \text{ is differentiable at } p$$

  - Moreover, in this case, such derivative is

$$\frac{\partial(\bar{p} \cdot \bar{x})}{\partial p} = \bar{x}$$

or in matrix notation

$$\nabla \mu_K(\bar{p}) = \bar{x}.$$

Appendix 2.2: Relationship between the Expenditure Function and Hicksian Demand

Proof I (Using Duality Theorem)

- The expenditure function is the support function for the set of all bundles in $\mathbb{R}^L_+$ for which utility reaches at least a level of u. That is,

$$\{x \in \mathbb{R}^L_+ : u(x) \geq u\}$$

  Using the Duality theorem, we can then state that there is a unique bundle in this set, $h(p, u)$, such that

$$p \cdot h(p, u) = e(p, u)$$

  where the right-hand side is the support function of this problem.

- Moreover, from the duality theorem, the derivative of the support function coincides with this unique bundle, i.e.,

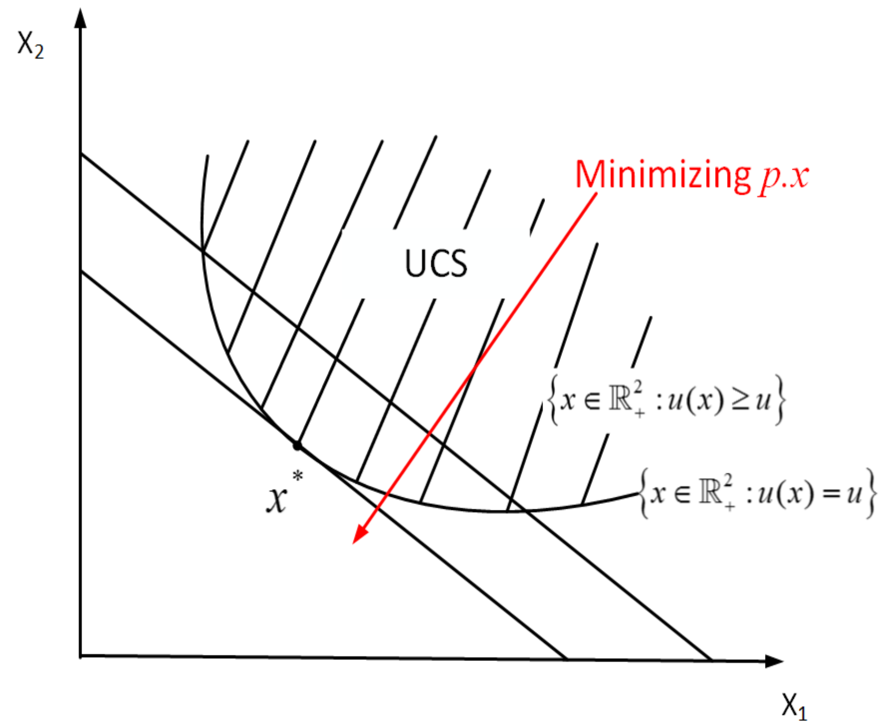
$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p, u) \text{ for every good } k$$

  or

$$\nabla_p e(p, u) = h(p, u)$$

Proof I (Using Duality Theorem)

- UCS is a convex and closed set.
- Hyperplane $p \cdot x^* = e(p, u)$ provides us with the minimal expenditure that still reaches utility level u.



Proof II (Using First Order Conditions)

- Totally differentiating the expenditure function $e(p,u)$,

$$\nabla_p e(p,u) = \nabla_p [p \cdot h(p,u)] = h(p,u) + \left[p \cdot D_p h(p,u)\right]^T$$

- And, from the FOCs in interior solutions of the EMP, we know that $p = \lambda \nabla u(h(p,u))$. Substituting,

$$\nabla_p e(p,u) = h(p,u) + \lambda[\nabla u(h(p,u)) \cdot D_p h(p,u)]^T$$

- But, since $\nabla u\big(h(p,u)\big) = u$ for all solutions of the EMP, then $\nabla u\big(h(p,u)\big) = 0$, which implies

$$\nabla_p e(p,u) = h(p,u)$$

That is, $\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u)$ for every good k.

Proof III (Using the Envelope Theorem)

- Using the envelope theorem in the expenditure function, we obtain

$$\frac{\partial e(p,u)}{\partial p_k} = \frac{\partial [p \cdot h(p,u)]}{\partial p_k} = h(p,u) + p\frac{\partial h(p,u)}{\partial p_k}$$

- And, since the Hicksian demand is already at the optimum, indirect effects are negligible, $\frac{\partial h(p,u)}{\partial p_k} = 0$, implying

$$\frac{\partial e(p,u)}{\partial p_k} = h(p,u)$$

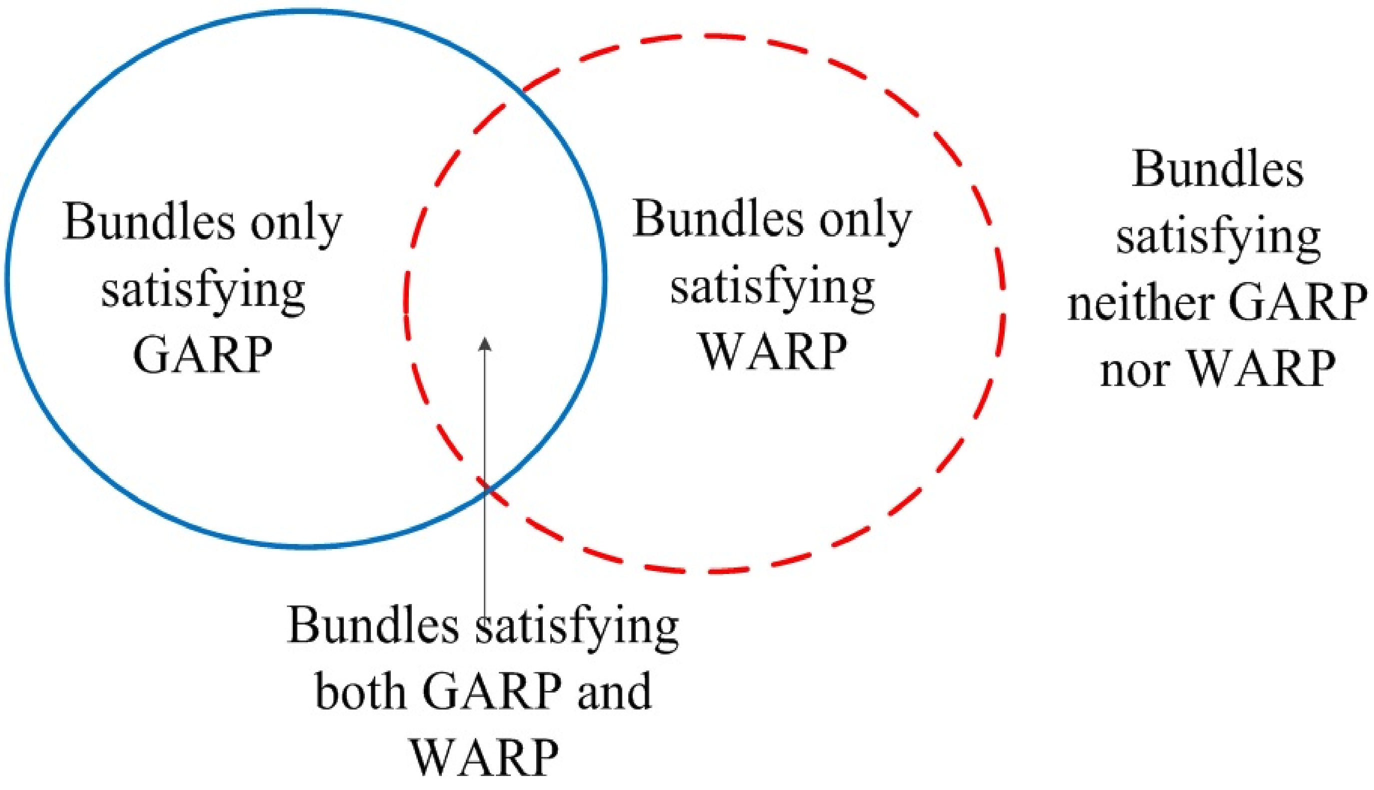
Appendix 2.3: Generalized Axiom of Revealed Preference (GARP)

GARP

- Consider a sequence of prices p^t where $t = 1,2, \dots, T$ with an associated sequence of chosen bundles x^t.
- ***GARP***: If bundle x^t is revealed preferred to x^{t+1} for all $t = 1,2, \dots, T$, i.e., $x^1 \gtrsim x^2, x^2 \gtrsim x^3, \dots, x^{T-1} \gtrsim x^T$, then x^T is *not* strictly revealed preferred to x^1, i.e., $x^T \nsucc x^1$.
  - More general axiom of revealed preference
  - Neither GARP nor WARP implies one another
  - Some choices satisfy GARP, some WARP, choices for which both axioms hold, and some for which none do.

GARP

- Choices satisfying GARP, WARP, both, or none



GARP

- ***Example A2.1*** (GARP holds, but WARP does not):
  - Consider the following sequence of price vectors and their corresponding demanded bundles

$$p_1 = (1,1,2) \quad x_1 = (1,0,0)$$
$$p_2 = (1,2,1) \quad x_2 = (0,1,0)$$
$$p_3 = (1,3,1) \quad x_3 = (0,0,2)$$

  - The change from $t = 1$ to $t = 2$ violates WARP, but not GARP.

GARP

- ***Example A2.1*** (continued):
  - Let us compare bundles x_1 with x_2.
  - For the premise of WARP to hold:
    - Since bundle x_1 is revealed preferred to x_2 in $t = 1$, i.e., $x_1 \succsim x_2$, it must be that $p_1 \cdot x_2 \leq w_1 = p_1 \cdot x_1$.
    - That is, bundle x_2 is affordable under bundle x_1's prices.
  - Substituting our values, we find that

$$(1 \times 0) + (1 \times 1) + (2 \times 0) = 1 \leq 1$$
$$= (1 \times 1) + (1 \times 0) + (2 \times 0)$$

  - Hence, the premise of WARP holds.

GARP

- ***Example A2.1*** (continued):
  - For WARP to be satisfied:
    - We must also have that $p_2 \cdot x_1 > w_2 = p_2 \cdot x_2$.
    - That is, bundle x_1 is unaffordable under the new prices and wealth.
  - Plugging our values yields

$$p_2 \cdot x_1 = (1 \times 1) + (2 \times 0) + (1 \times 0) = 1$$
$$p_2 \cdot x_2 = w_2 = (1 \times 0) + (2 \times 1) + (1 \times 0) = 2$$

  - That is, $p_2 \cdot x_1 < w_2$, i.e., bundle x_1 is still affordable under period two's prices.
  - Thus WARP is violated.

GARP

- ***Example A2.1*** (continued):
  - Let us check GARP.
  - Assume that bundle x_1 is revealed preferred to x_2, $x_1 \succsim x_2$, and that x_2 is revealed preferred to x_3, $x_2 \succsim x_3$.
  - It is easy to show that $x_3 \nsucc x_1$ as bundle x_3 is not affordable under bundle x_1's prices.
  - Hence, x_3 cannot be revealed preferred to x_1,
  - Thus, GARP is not violated.

GARP

- ***Example A2.2*** (WARP holds, GARP does not):
  - Consider the following sequence of price vectors and their corresponding demanded bundles

$$p_1 = (1,1,2) \quad x_1 = (1,0,0)$$
$$p_2 = (2,1,1) \quad x_2 = (0,1,0)$$
$$p_3 = (1,2,1+\varepsilon) \quad x_3 = (0,0,1)$$

GARP

- ***Example A2.2*** (continued):
  - For the premise of WARP to hold:
    - Since bundle x_1 is revealed preferred to x_2 in $t = 1$, i.e., $x_1 \succsim x_2$, it must be that $p_1 \cdot x_2 \leq w_1 = p_1 \cdot x_1$.
    - That is, bundle x_2 is affordable under bundle x_1's prices.
  - Substituting our values, we find that

$$(1 \times 0) + (1 \times 1) + (2 \times 0) = 1 \leq 1$$
$$= (1 \times 1) + (1 \times 0) + (2 \times 0)$$

  - Hence, the premise of WARP holds.

GARP

- ***Example A2.2*** (continued):
  - For WARP to be satisfied:
    - We must also have that $p_2 \cdot x_1 > w_2 = p_2 \cdot x_2$.
    - That is, bundle x_1 is unaffordable under the new prices and wealth.
  - Plugging our values yields

$$(2 \times 1) + (1 \times 0) + (1 \times 0) = 2 > 1$$
$$= (2 \times 0) + (1 \times 1) + (1 \times 0)$$

  - Thus, WARP is satisfied.

GARP

- ***Example A2.2*** (continued):
  - A similar argument applies to the comparison of the choices in $t = 2$ and $t = 3$.
  - Bundle x_3 is affordable under the prices at $t = 2$,
  $$p_2 \cdot x_3 = 1 \leq 1 = w_2 = p_2 \cdot x_2$$
  - But x_2 is unaffordable under the prices of $t = 3$, i.e., $p_3 \cdot x_2 > w_3 = p_3 \cdot x_3$, since
  $$(1 \times 0) + (2 \times 1) + (1 + \varepsilon \times 0) = 2 >$$
  $$(1 \times 0) + (2 \times 0) + (1 + \varepsilon \times 1) = 1 + \varepsilon$$
  - Hence, WARP also holds in this case.

GARP

- ***Example A2.2*** (continued):
  - Furthermore, bundle x_3 is unaffordable under bundle x_1's prices, i.e., $p_1 \cdot x_3 \nleq w_1 = p_1 \cdot x_1$,

$$p_1 \cdot x_3 = (1 \times 0) + (1 \times 0) + (2 \times 1) = 2$$
$$p_1 \cdot x_1 = (1 \times 1) + (1 \times 0) + (2 \times 0) = 1$$

  - Thus, the premise for WARP does not hold when comparing bundles x_1 and x_3.
  - Hence, WARP is not violated.

GARP

- ***Example A2.2*** (continued):
  - Let us check GARP.
  - Assume that bundle x_1 is revealed preferred to x_2, and that x_2 is revealed preferred to x_3, i.e., $x_1 \succsim x_2$ and $x_2 \succsim x_3$.
  - Comparing bundles x_1 and x_3, we can see that bundle x_1 is affordable under bundle x_3's prices, that is

$$p_3 \cdot x_1 = (1 \times 1) + (2 \times 0) + (1 + \varepsilon \times 0) = 1$$
$$\ngtr 1 + \varepsilon = w_3 = p_3 \cdot x_3$$

GARP

- ***Example A2.2*** (continued):
  - In other words, both x_1 and x_3 are affordable at $t = 3$ but only x_3 is chosen.
  - Then, the consumer is revealing a preference for x_3 over x_1, i.e., $x_3 \succsim x_1$.
  - This violates GARP.

GARP

- ***Example A2.3*** (Both WARP and GARP hold):
  - Consider the following sequence of price vectors and their corresponding demanded bundles

$$p_1 = (1,1,2) \quad x_1 = (1,0,0)$$
$$p_2 = (1,2,1) \quad x_2 = (0,1,0)$$
$$p_3 = (3,2,1) \quad x_3 = (0,0,1)$$

GARP

- ***Example A2.3*** (continued):
  - Let us check WARP.
  - Note that bundle x_2 is affordable under the prices at $t = 1$, i.e., $p_1 \cdot x_2 \leq w_1 = p_1 \cdot x_1$, since

    $$(1 \times 0) + (1 \times 1) + (1 \times 0) = 1 \leq 1$$
    $$= (1 \times 1) + (1 \times 0) + (1 \times 0)$$

  - However, bundle x_1 is unaffordable under the prices at $t = 2$, i.e., $p_2 \cdot x_1 > w_2 = p_2 \cdot x_2$, since

    $$(2 \times 1) + (1 \times 0) + (1 \times 0) = 2 > 1$$
    $$= (2 \times 0) + (1 \times 1) + (1 \times 0)$$

  - Thus WARP is satisfied.

GARP

- ***Example A2.3*** (continued):
  - A similar argument applies to the comparison of the choices in $t = 2$ and $t = 3$.
  - Bundle x_3 is affordable under the prices at $t = 2$,
  $$p_2 \cdot x_3 = 1 \leq 1 = w_2 = p_2 \cdot x_2$$
  - But x_2 is unaffordable under the prices of $t = 3$, i.e., $p_3 \cdot x_2 > w_3 = p_3 \cdot x_3$, since
  $$(3 \times 0) + (2 \times 1) + (1 \times 0) = 2 >$$
  $$(3 \times 0) + (2 \times 0) + (1 \times 1) = 1 + \varepsilon$$
  - Thus, WARP also holds in this case.

GARP

- ***Example A2.3*** (continued):
  - A similar argument applies to the comparison of the choices in $t = 1$ and $t = 3$.
  - Bundle x_3 is affordable under the prices at $t = 1$, i.e., $p_1 \cdot x_3 \leq w_1 = p_1 \cdot x_1$, since
$$p_1 \cdot x_3 = 1 \leq 1 = w_2 = p_1 \cdot x_1$$
  - But bundle x_1 is unaffordable under the prices of $t = 3$, i.e., $p_3 \cdot x_1 > w_3 = p_3 \cdot x_3$, since
$$p_3 \cdot x_1 = (3 \times 1) + (2 \times 0) + (1 \times 0) = 3$$
$$p_3 \cdot x_3 = (3 \times 0) + (2 \times 0) + (1 \times 1) = 1$$
  - Hence, WARP is also satisfied

GARP

- ***Example A2.3*** (continued):
  - Let us check GARP.
  - We showed above that x_1 is unaffordable under bundle x_3's prices, i.e., $p_3 \cdot x_1 > w_3 = p_3 \cdot x_3$.
  - We cannot establish bundle x_3 being strictly preferred to bundle x_1, i.e., $x_3 \nsucc x_1$.
  - Hence, GARP is satisfied.

GARP

- ***Example A2.4*** (Neither WARP nor GARP hold):
  - Consider the following sequence of price vectors and their corresponding demanded bundles

$$p_1 = (1,1,1) \quad x_1 = (1,0,0)$$
$$p_2 = (2,1,1) \quad x_2 = (0,1,0)$$
$$p_3 = (1,2,1+\varepsilon) \quad x_3 = (0,0,1)$$

  - This is actually a combination of the first two examples.

GARP

- ***Example A2.4*** (continued):
  - At $t = 1$, WARP will be violated by the same method as in our first example.
  - At $t = 3$, GARP will be violated by the same method as in our second example.
  - Hence, neither WARP nor GARP hold in this case.

GARP

- GARP constitutes a sufficient condition for utility maximization.
- That is, if the sequence of price-bundle pairs (p^t, x^t) satisfies GARP, then it must originate from a utility maximizing consumer.
- We refer to (p^t, x^t) as a set of "data."

GARP

- ***Afriat's theorem***. For a sequence of price-bundle pairs (p^t, x^t), the following statements are equivalent:
  1) The data satisfies GARP.
  2) The data can be rationalized by a utility function satisfying LNS.
  3) There exists positive numbers (u^t, λ^t) for all $t = 1,2, \dots, T$ that satisfies the Afriat inequalities

$$u^s \leq u^t + \lambda^t p^t (x^s - x^t) \text{ for all } t, s$$

  4) The data can rationalized by a continuous, concave, and strongly monotone utility function.

GARP

- A data set is "rationalized" by a utility function if:
  - for every pair (p^t, x^t), bundle x^t yields a higher utility than any other feasible bundle x, i.e., $u(x^t) \geq u(x)$ for all x in budget set $B(p^t, p^t x^t)$.
- Hence, if a data set satisfies GARP, there exists a well-behaved utility function rationalizing such data.
  - That is, the utility function satisfies LNS, continuity, concavity, and strong monotonicity.

GARP

- Condition (3) in Afriat's Theorem has a concavity interpretation:
  - From FOCs, we have $Du(x^t) = \lambda^t p^t$.
  - By concavity,
$$\frac{u(x^t)-u(x^s)}{x^t-x^s} \leq u'(x^s)$$
  - Or, re-arranging,
$$u(x^t) \leq u(x^s) + u'(x^s)(x^t - x^s)$$
  - Since $u'(x^s) = \lambda^s p^s$, it can be expressed as Afriat's inequality
$$u(x^t) \leq u(x^s) + \lambda^s p^s(x^t - x^s)$$

GARP

- Concave utility function
  - The utility function at point x^s is steeper than the ray connecting points *A* and *B*.